

# Filtered expansions in general relativity and one BKL-bounce

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**Abstract:** When the vacuum Einstein equations are formulated in terms of a frame, rather than a metric, can one perturb solutions with a degenerate frame into ones with a nondegenerate frame? In examples we point out that one can encounter issues already at the level of formal perturbative expansions; namely the cohomological, so-called space of obstructions is nonzero. In this paper we propose a perturbative expansion based on filtrations. We construct and prove properties of a specific filtration, intended to make mathematical sense of one BKL-bounce, a building block of a well-known but very heuristic conjecture due to Belinskii, Khalatnikov and Lifshitz (which would involve sticking together an infinite sequence of single bounces). It seems possible that now the space of obstructions is zero, but this question is left open.

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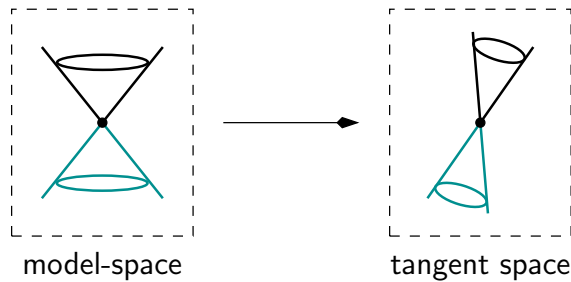
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## 1 Introduction

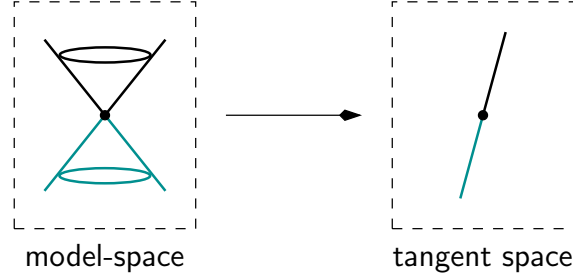
The vacuum Einstein equations of general relativity are traditionally formulated in terms of a metric, but they can also be formulated in terms of a frame. It is then tempting to start from a solution with a degenerate frame and to try to perturb it into one with a nondegenerate frame. The rationale is that degenerate objects tend to be simpler than nondegenerate ones. We refer to this kind of perturbation as a degenerate-to-nondegenerate perturbation.

What is a degenerate frame? Suppose we are on a manifold diffeomorphic to  $\mathbb{R}^4$ . Then a frame is, over each point of the manifold, an invertible linear map



where the fixed model-space<sup>3</sup> has a fixed inner product or conformal inner product of signature  $-+++$ . We would more specifically call this a nondegenerate frame. By contrast, a degenerate frame is a linear map that is not invertible:

<sup>3</sup>Model-space: fiber of a real vector bundle of rank 4, over the 4-dim manifold.



A solution with nondegenerate frame yields a metric that solves the vacuum Einstein equations (i.e. a metric with vanishing Ricci curvature), whereas one with degenerate frame does not have an associated metric.

We do not actually carry out a degenerate-to-nondegenerate perturbation in this paper. Below we indicate some of the issues that one encounters. We then make a proposal based on filtrations, the main topic of this paper.

## 1.1 Language

At first it is enough to know just the bare algebraic structure of the formalism that we use; more details are introduced as needed. Let  $\mathcal{E} = \bigoplus_{k \in \mathbb{Z}} \mathcal{E}^k$  be a real graded Lie algebra<sup>4</sup>. A natural object in a graded Lie algebra is

$$\text{Sol}(\mathcal{E}) = \{\gamma \in \mathcal{E}^1 \mid [\gamma, \gamma] = 0\}$$

Conceptually the quotient  $\text{Sol}(\mathcal{E})/(\text{automorphisms generated by } \mathcal{E}^0)$  is deemed the more basic object, but it is ill-defined at this general level since it presumes that we can exponentiate elements of the Lie algebra  $\mathcal{E}^0$ .

General relativity is of this form: one can choose  $\mathcal{E}$  so that  $\text{Sol}(\mathcal{E})$  is the set of solutions to the vacuum Einstein equations [RT]. Here  $\mathcal{E} = \mathcal{E}^0 \oplus \dots \oplus \mathcal{E}^4$ . Linearly associated to the unknown  $\gamma \in \mathcal{E}^1$  is a frame<sup>5</sup>. Essentially,  $\mathcal{E}^0$  generates self-diffeomorphisms of the 4-dim manifold and rotations of model-space.

<sup>4</sup>A real graded Lie algebra (see Nijenhuis, Richardson [NR]) is a  $\mathbb{Z}$ -graded real vector space with an  $\mathbb{R}$ -bilinear bracket  $[\cdot, \cdot] : \mathcal{E}^k \times \mathcal{E}^\ell \rightarrow \mathcal{E}^{k+\ell}$  such that  $[x, y] = -(-1)^{|x||y|}[y, x]$  and  $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0$  if  $x \in \mathcal{E}^{|x|}$ ,  $y \in \mathcal{E}^{|y|}$ ,  $z \in \mathcal{E}^{|z|}$ .

<sup>5</sup>Associated to  $\gamma$  is also an affine connection, and essentially the frame and the connection determine  $\gamma$ . However the map  $\gamma \mapsto \text{connection}$  is nonlinear, and is not defined (is singular) when the frame is degenerate. It would therefore be misleading and wrong, especially in this paper, to say that  $\gamma$  consists of a frame and an affine connection.

## 1.2 Issues with naive degenerate-to-nondegenerate perturbations

Going back to degenerate-to-nondegenerate perturbations, the most naive proposal is to perturb the zero solution  $0 \in \text{Sol}(\mathcal{E})$ . The associated frame vanishes, that is, the frame is fully degenerate. While perturbations certainly exist<sup>6</sup>, they cannot obviously be obtained from perturbation theory about zero. The problem is that the derivative of  $\mathcal{E}^1 \rightarrow \mathcal{E}^2, \gamma \mapsto [\gamma, \gamma]$  vanishes at zero.

A more serious proposal is to perturb a  $\gamma_{(0)} \in \text{Sol}(\mathcal{E})$  whose frame satisfies, at each point of the 4-dim manifold, the equivalent conditions:

- The image of the past cone is disjoint from the image of the future cone.
- The kernel is a spacelike subspace of model-space.

These are natural conditions to impose and one could guess that they suffice to do perturbation theory<sup>7</sup> about  $\gamma_{(0)}$ , but this is not obviously the case.

The derivative of  $\mathcal{E}^1 \rightarrow \mathcal{E}^2, \gamma \mapsto [\gamma, \gamma]$  at  $\gamma_{(0)}$  is two times the  $k = 1$  instance of the map  $d_{(0)} = [\gamma_{(0)}, \cdot] : \mathcal{E}^k \rightarrow \mathcal{E}^{k+1}$ , but it is good to define this map for all  $k$  right away. The graded Lie algebra axioms imply<sup>8</sup> that this is a differential,  $(d_{(0)})^2 = 0$ . Its cohomologies play an important role in perturbation theory, see for example Gerstenhaber [G]. In particular, the 2nd cohomology

$$\ker(d_{(0)}|_{\mathcal{E}^2}) / \text{image}(d_{(0)}|_{\mathcal{E}^1})$$

is called the space of obstructions. If this 2nd cohomology is zero, then every finite-order perturbation extends to one of all orders, i.e. to a formal power series perturbation, by a standard argument.

In this paper we do not consider all such  $\gamma_{(0)}$ , only a certain class. For this class, the 2nd cohomology of  $d_{(0)}$  is nonzero as shown in §15.4. Perturbations

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<sup>6</sup>Take the ray through any nontrivial element of  $\text{Sol}(\mathcal{E})$ , say Minkowski spacetime.

<sup>7</sup>Consider the toy example

$$\begin{pmatrix} \partial_0 + \varepsilon \partial_3 & \varepsilon \partial_1 + i\varepsilon \partial_2 \\ \varepsilon \partial_1 - i\varepsilon \partial_2 & \partial_0 - \varepsilon \partial_3 \end{pmatrix}$$

where  $\partial_0, \partial_1, \partial_2, \partial_3$  are partial derivatives,  $\varepsilon \in \mathbb{R}$  and  $i = \sqrt{-1}$ . If  $\varepsilon \neq 0$  then the matrix entries make a (complex) nondegenerate frame on  $\mathbb{R}^4$ , and the matrix is symmetric hyperbolic. If  $\varepsilon = 0$  then the frame is degenerate, but the matrix is symmetric hyperbolic nonetheless. Beware that this is a poor toy example for a gauge theory like general relativity.

<sup>8</sup>In a graded Lie algebra,  $[x, [x, \cdot]] = \frac{1}{2}[[x, x], \cdot]$  for all elements  $x$  of degree one.

may well exist, but they cannot obviously be obtained from perturbation theory about  $\gamma_{(0)}$ , much like for the zero solution  $0 \in \text{Sol}(\mathcal{E})$ .

We emphasize that these are issues encountered at the level of formal power series perturbations, before even broaching the issue of convergence.

### 1.3 Goal of this paper and open questions

In this paper we construct a filtration of  $\mathcal{E}$  that can be used to set up a filtered expansion, a refinement of the above  $\gamma_{(0)}$ -proposal. We call  $\gamma_{(0)}$  the naive leading term. It is only a piece of an object  $\gamma_0 = \gamma_{(0)} \oplus \dots$  that we call without reservation the leading term; the direct sum is defined later in this introduction. Whereas  $\gamma_{(0)}$  has a degenerate frame,  $\gamma_0$  already describes the infinitesimal opening-up to a nondegenerate frame.

This  $\gamma_0$  is an element not of  $\mathcal{E}$ , but of another graded Lie algebra that has the same ‘size’ as  $\mathcal{E}$ , and that is defined using the filtration.

This  $\gamma_0$  defines a differential  $d_0$  whose 2nd cohomology is the new space of obstructions. We do not prove that this space is zero, but it seems possible that it is. The 1st cohomology is the space of non-equivalent solutions to the linearized equations. It would be interesting to calculate these cohomologies. If things pan out, one would have a concrete mathematical object for the heuristic idea of one BKL-bounce, at the preliminary level of formal power series perturbations<sup>9</sup>.

By one BKL-bounce we mean the basic building block of an inspiring but very heuristic conjecture due to Belinskii, Khalatnikov, Lifshitz [LK, BKL], which would involve sticking together (cf. §16.16) an infinite sequence of bounces. Here we do not even attempt to summarize this conjecture in words; we have used the idea of the degenerate-to-nondegenerate perturbation as the pedagogical motivation for this paper, because it can be read and understood on its own.

Technically, what we do in this paper is to construct a filtration that:

- Encodes a class of  $\gamma_0 = \gamma_{(0)} \oplus \dots$  (§1.9 and §1.10).
- Imposes a lower triangular structure on perturbation theory (§1.7).
- Is by design amenable to gauge-fixing to hyperbolic equations, even though the construction of the filtration does not involve any gauge-fixing (§1.13).

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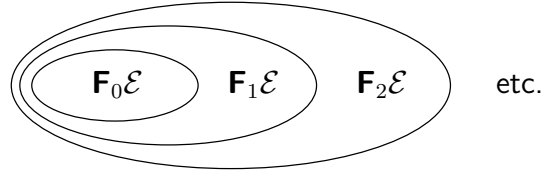
<sup>9</sup>It would be very interesting to look systematically for other interesting filtrations of  $\mathcal{E}$ .

Choosing filtrations involves conceptual issues, specific to general relativity.

The primary objects in this paper are filtrations. By contrast,  $\gamma_0 = \gamma_{(0)} \oplus \dots$  only appears in this introduction and again briefly at the end of this paper, where we write down explicitly the equations that  $\gamma_0$  has to satisfy (§16).

#### 1.4 What is a filtration?

A filtration is an increasing sequence of subspaces of  $\mathcal{E}$ :



The basic algebraic requirements for a filtration indexed by  $p \in \mathbb{Z}_{\geq 0}$  are:

- $\mathbf{F}_p \mathcal{E} = \bigoplus_k \mathbf{F}_p \mathcal{E}^k$  where  $\mathbf{F}_p \mathcal{E}^k \subseteq \mathcal{E}^k$  are linear subspaces<sup>10</sup>.
- $\mathbf{F}_p \mathcal{E} \subseteq \mathbf{F}_{p+1} \mathcal{E}$ .
- $[\mathbf{F}_p \mathcal{E}, \mathbf{F}_q \mathcal{E}] \subseteq \mathbf{F}_{p+q} \mathcal{E}$ .
- $\exists p : \mathbf{F}_p \mathcal{E} = \mathcal{E}$ .

That is, the filtration is by graded subspaces; is increasing; respects the bracket; exhausts<sup>11</sup> at a finite  $p$ . Note in particular that  $\mathbf{F}_0 \mathcal{E}$  is a subalgebra.

To appreciate these abstract conditions, it is good to recall some of the various roles of the graded Lie algebra bracket  $[\cdot, \cdot] : \mathcal{E}^k \times \mathcal{E}^\ell \rightarrow \mathcal{E}^{k+\ell}$ :

$\mathcal{E}^0 \times \mathcal{E}^0 \rightarrow \mathcal{E}^0$	bracket of the ‘infinitesimal gauge group’ Lie algebra $\mathcal{E}^0$
$\mathcal{E}^0 \times \mathcal{E}^k \rightarrow \mathcal{E}^k$	action of the infinitesimal gauge group on $\mathcal{E}$
$\mathcal{E}^1 \times \mathcal{E}^1 \rightarrow \mathcal{E}^2$	used to state the vacuum Einstein equations
$\mathcal{E}^1 \times \mathcal{E}^2 \rightarrow \mathcal{E}^3$	used to state the key identity $\forall \gamma \in \mathcal{E}^1 : [\gamma, [\gamma, \gamma]] = 0$

<sup>10</sup>Vector subspaces, or maybe even submodules over the ring of smooth real functions on the 4-dim manifold. All actual constructions in this paper yield submodules.

<sup>11</sup> To be exhaustive simply means  $\bigcup_p \mathbf{F}_p \mathcal{E} = \mathcal{E}$ . We use the stronger condition  $\exists p : \mathbf{F}_p \mathcal{E} = \mathcal{E}$  since it is more concrete, and it suffices for this paper.

## 1.5 Filtered expansions and the Rees algebra

The basic idea is this: One looks for formal power series solutions in a variable  $s$ , with the coefficient of  $s^p$  drawn from  $\mathbf{F}_p \mathcal{E}^1$ . That is, the filtration specifies at what stage  $p$  the different degrees of freedom can kick in.

To implement this idea, introduce the so-called Rees algebra

$$\mathcal{P} = \bigoplus_{p \geq 0} s^p \mathbf{F}_p \mathcal{E}$$

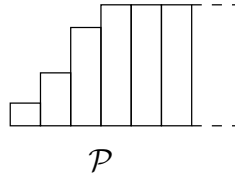
understood as a subspace of  $\mathcal{E}[[s]]$ , the formal power series in  $s$  with coefficients in  $\mathcal{E}$ . Note that  $\mathcal{E}[[s]]$  is a graded Lie algebra over  $\mathbb{R}[[s]]$ .

The first three bullets in §1.4 can now be restated as:

- $\mathcal{P} = \bigoplus_k \mathcal{P}^k$  where  $\mathcal{P}^k = \bigoplus_{p \geq 0} s^p \mathbf{F}_p \mathcal{E}^k$ .
- $\mathcal{P} \rightarrow \mathcal{P}, x \mapsto sx$  is an injective map.
- $\mathcal{P}$  is itself a graded Lie algebra over  $\mathbb{R}[[s]]$ , a subalgebra of  $\mathcal{E}[[s]]$ .

The set of formal power series solution to the vacuum Einstein equations, filtered by  $\mathbf{F}$ , is then concisely given by  $\text{Sol}(\mathcal{P}) = \{\gamma \in \mathcal{P}^1 \mid [\gamma, \gamma] = 0\}$  where the bracket is understood to be the bracket in  $\mathcal{P}$ .

We informally picture the Rees algebra as

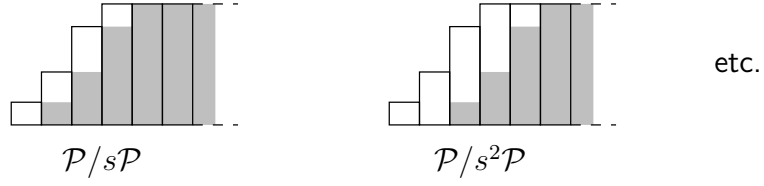


and in drawing the picture we have assumed that  $\mathbf{F}_p \mathcal{E} = \mathcal{E}$  if and only if  $p \geq 3$ .

## 1.6 The role of $\mathcal{P}/s\mathcal{P}$

The quotient  $\mathcal{P}/s\mathcal{P}$ , a standard object in algebra, plays an outstanding role for filtered expansions. It has the same ‘size’ as  $\mathcal{E}$ .

Suppose we want to construct a  $\gamma \in \text{Sol}(\mathcal{P})$ . One can try to break the problem into pieces and solve  $[\gamma, \gamma] = 0$  first modulo  $s\mathcal{P}$ , then modulo  $s^2\mathcal{P}$ , etc. In the following pictures, the darker parts are divided out:



The  $s\mathcal{P} \supseteq s^2\mathcal{P} \supseteq \dots$  are ideals,  $[\mathcal{P}, s^p\mathcal{P}] \subseteq s^p\mathcal{P}$ . Hence one has a sequence of graded Lie algebra morphisms  $0 \leftarrow \mathcal{P}/s\mathcal{P} \leftarrow \mathcal{P}/s^2\mathcal{P} \leftarrow \dots$  whereby:

- Each of these morphisms is surjective.  
Each of these morphisms has kernel canonically isomorphic<sup>12</sup> to  $\mathcal{P}/s\mathcal{P}$ .
- Each  $x \in \mathcal{P}$  corresponds to a sequence of elements  $0 \leftarrow x_0 \leftarrow x_1 \leftarrow \dots$  and in particular if  $\gamma \in \mathcal{P}^1$  corresponds to  $0 \leftarrow \gamma_0 \leftarrow \gamma_1 \leftarrow \dots$  then

$$\gamma \in \text{Sol}(\mathcal{P}) \quad \Longleftrightarrow \quad \forall p : \gamma_p \in \text{Sol}(\mathcal{P}/s^{p+1}\mathcal{P})$$

When one tries to construct the  $\gamma_p$  in succession, the quotient  $\mathcal{P}/s\mathcal{P}$  plays an outstanding role as the kernel of the morphisms; because of the leading term  $\gamma_0 \in \text{Sol}(\mathcal{P}/s\mathcal{P})$ ; and because of the differential  $d_0$  defined below.

Explicitly

$$\mathcal{P}/s\mathcal{P} = \bigoplus_{p \geq 0} s^p(\mathbf{F}_p\mathcal{E}/\mathbf{F}_{p-1}\mathcal{E})$$

It is a graded Lie algebra relative to the  $k$ -grading, whose bracket respects both the  $p$ -grading and the  $k$ -grading. The  $p$ -grading has only finitely many nonzero components because the filtration exhausts at a finite  $p$ .

## 1.7 The lower triangular differential $d_0$

Let  $\gamma_0 \in \text{Sol}(\mathcal{P}/s\mathcal{P})$ , the leading term<sup>13</sup>. Define

$$d_0 = [\gamma_0, \cdot] : \mathcal{P}/s\mathcal{P} \rightarrow \mathcal{P}/s\mathcal{P}$$

<sup>12</sup>  $\ker(\mathcal{P}/s^p\mathcal{P} \leftarrow \mathcal{P}/s^{p+1}\mathcal{P}) = s^p\mathcal{P}/s^{p+1}\mathcal{P}$  with canonical isomorphism  $\leftarrow \mathcal{P}/s\mathcal{P}$ ,  $s^p x \leftarrow x$ .

<sup>13</sup>At the level of formal power series, the basic object of study is the quotient

$$\{\gamma \in \text{Sol}(\mathcal{P}) \mid \gamma = \gamma_0 \bmod s\mathcal{P}^1\} / \exp([s\mathcal{P}^0, \cdot])$$

The quotient is in the sense of a group action. The denominator is the group associated to the Lie algebra  $s\mathcal{P}^0$  by the Baker-Campbell-Hausdorff formula.



which is a differential,  $(d_0)^2 = 0$ . The role of the cohomologies of  $d_0$  has been alluded to in §1.2, §1.3. This differential raises the  $k$ -grading by one:

$$0 \xrightarrow{d_0} \mathcal{P}^0/s\mathcal{P}^0 \xrightarrow{d_0} \mathcal{P}^1/s\mathcal{P}^1 \xrightarrow{d_0} \dots \xrightarrow{d_0} \mathcal{P}^4/s\mathcal{P}^4 \xrightarrow{d_0} 0$$

In the  $p$ -grading it is lower-triangular:

$$d_0 = \begin{pmatrix} d_{(0)} & 0 & 0 & \dots \\ * & d_{(0)} & 0 & \dots \\ * & * & d_{(0)} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This is the block decomposition of  $d_0$  by the  $p$ -grading of  $\mathcal{P}/s\mathcal{P}$ . The rows and columns are indexed by  $p \geq 0$ , and there are only finitely many of them.

The  $p$ -th entry on the main diagonal is the differential

$$d_{(0)} = [\gamma_{(0)}, \cdot] : \mathbf{F}_p\mathcal{E}/\mathbf{F}_{p-1}\mathcal{E} \rightarrow \mathbf{F}_p\mathcal{E}/\mathbf{F}_{p-1}\mathcal{E}$$

where  $\gamma_{(0)} \in \text{Sol}(\mathbf{F}_0\mathcal{E})$  is the first direct summand of  $\gamma_0 = \gamma_{(0)} \oplus \dots$  and is called the naive leading term. Hence each entry of the main diagonal is a complex:

$$0 \xrightarrow{d_{(0)}} \mathbf{F}_p\mathcal{E}^0/\mathbf{F}_{p-1}\mathcal{E}^0 \xrightarrow{d_{(0)}} \mathbf{F}_p\mathcal{E}^1/\mathbf{F}_{p-1}\mathcal{E}^1 \xrightarrow{d_{(0)}} \dots \xrightarrow{d_{(0)}} \mathbf{F}_p\mathcal{E}^4/\mathbf{F}_{p-1}\mathcal{E}^4 \xrightarrow{d_{(0)}} 0$$

By contrast, the entries of the subdiagonals are determined by the remaining summands in  $\gamma_0 = \gamma_{(0)} \oplus \dots$  and do not individually correspond to complexes<sup>14,15</sup>.

## 1.8 Notational overview

In this paper we first construct a specific  $\mathbb{Z}_{\geq 0}$ -indexed filtration  $\mathbf{F}$ . We then take two copies of this filtration,  $\mathbf{F}'$  and  $\mathbf{F}''$ , that differ only by a relative rotation of model-space. We then define a  $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ -indexed filtration:

$$\mathbf{f}_{p'p''}\mathcal{E} = \mathbf{F}'_{p'}\mathcal{E} \cap \mathbf{F}''_{p''}\mathcal{E}$$

This is the filtration intended for one BKL-bounce:

<sup>14</sup>As a toy example, let  $m = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$  where  $a, c$  are real square matrices,  $b$  a real rectangular matrix, and  $m^2 = 0$ . Then  $a^2 = ba + cb = c^2 = 0$ . Hence  $b$  descends to a map  $H(a) \rightarrow H(c)$  where  $H(a) = \ker a / \text{image } a$ . There is a canonical vector space isomorphism

$$H(m)/X \oplus X \simeq H(0 \rightarrow H(a) \xrightarrow{b} H(c) \rightarrow 0)$$

where  $X \subseteq H(m)$  is the subspace of elements that have a representative of the form  $\begin{pmatrix} 0 \\ * \end{pmatrix}$ . This is a spectral sequence [C]. Note that  $H(m) \simeq H(m)/X \oplus X$  but non-canonically so.

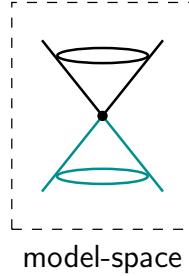
<sup>15</sup>The subdiagonals may play a crucial role for the filtration  $\mathbf{f}$  constructed in this paper. An observation in §16.15 suggests that there are degeneracies in the diagonal entries that may be resolved by the subdiagonals. See also the closely related §15.4.

$$\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & \\
\mathbf{f}_{02}\mathcal{E} & \subseteq & \mathbf{f}_{12}\mathcal{E} & \subseteq & \mathbf{f}_{22}\mathcal{E} & \cdots & \\
\cup & & \cup & & \cup & & \\
\mathbf{f}_{01}\mathcal{E} & \subseteq & \mathbf{f}_{11}\mathcal{E} & \subseteq & \mathbf{f}_{21}\mathcal{E} & \cdots & \\
\cup & & \cup & & \cup & & \\
\mathbf{f}_{00}\mathcal{E} & \subseteq & \mathbf{f}_{10}\mathcal{E} & \subseteq & \mathbf{f}_{20}\mathcal{E} & \cdots & 
\end{array}$$

While filtered expansions based on  $\mathbf{F}$  involve a single formal variable  $s$ , those based on  $\mathbf{f}$  involve two formal variables  $s', s''$ . While the naive leading term is a  $\gamma_{(0)} \in \text{Sol}(\mathbf{F}_0\mathcal{E})$  in one case, it is a  $\gamma_{(0)} \in \text{Sol}(\mathbf{f}_{00}\mathcal{E})$  in the other. And so forth. Modulo such modifications, the previous discussion stays intact.

### 1.9 Degeneracy of the frame encoded in $\mathbf{F}_0$ and $\mathbf{f}_{00}$

Recall that a frame is, over each point of the 4-dim manifold, a linear map with domain a fixed model-space with fixed conformal inner product<sup>16</sup>:



The filtration  $\mathbf{F}$  constructed in this paper has the following property:

- There is a fixed 2-dim spacelike subspace of model-space that is contained in the kernel of the frame of every element of  $\mathbf{F}_0\mathcal{E}^1$ .

The copies  $\mathbf{F}'$ ,  $\mathbf{F}''$  are rotated relative to each other such that the associated pair of 2-dim subspaces span only a 3-dim spacelike subspace:

- There is a fixed 3-dim spacelike subspace of model-space that is contained in the kernel of the frame of every element of  $\mathbf{f}_{00}\mathcal{E}^1$ .

<sup>16</sup>This model-space is the dual of what is called  $(V\bar{V})_{\text{real}}$  in [RT] and in this paper.

Hence the frame of each  $\gamma_{(0)} \in \mathbf{f}_{00}\mathcal{E}^1$  has rank 1, if no extra degeneracy occurs. The associated tangent space direction is  $\gamma_{(0)}$ -dependent, not enshrined in  $\mathbf{f}_{00}\mathcal{E}^1$ .

This discussion gives some geometric intuition for  $\mathbf{F}$  and  $\mathbf{f}$ , but it is not a full description of these filtrations.

## 1.10 Relation to BKL

The relation to BKL is most apparent in §15 where we point out that, up to innocuous simplifications also discussed there,  $\gamma_{(0)} \in \text{Sol}(\mathbf{f}_{00}\mathcal{E})$  if and only if

$$\begin{aligned}\xi_a(a_1) &= (a_6)^2 \\ \xi_a(a_2) &= -(a_6)^2 \\ \xi_a(a_3) &= (a_6)^2 \\ \xi_a(a_6) &= a_2a_6 \\ 0 &= a_2a_3 + a_3a_1 + a_1a_2 - (a_6)^2\end{aligned}$$

Here  $\gamma_{(0)}$  is parametrized by a vector field  $\xi_a$  (the frame) and by real functions  $a_1, a_2, a_3, a_6$  on the 4-dim manifold; the details are in §15.

These are four ordinary differential equations along  $\xi_a$ , and one equation without derivatives. Suppose for concreteness that we are on  $\mathbb{R}^4$  and that  $\xi_a$  is the first partial derivative. Then  $a_6$  goes to zero along each integral curve of  $\xi_a$ , both towards the past and future. The other three functions approach past limits  $a^{-\infty}$  and future limits  $a^{+\infty}$  that can differ from integral curve to integral curve. These limits satisfy

$$0 = a_2^{\pm\infty}a_3^{\pm\infty} + a_3^{\pm\infty}a_1^{\pm\infty} + a_1^{\pm\infty}a_2^{\pm\infty}$$

On integral curves on which  $a_6$  is not identically zero, one has  $a_2^{-\infty} > 0$  and<sup>17</sup>

$$a^{+\infty} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{pmatrix} a^{-\infty}$$

This  $\xi_a$  is the only direction of differentiation of the first order operator  $d_{(0)}$ , but this is not true for entries of the subdiagonals of the matrix of  $d_0$  in §1.7.

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<sup>17</sup>The quotients  $p_i^{\pm\infty} = a_i^{\pm\infty}/(a_1^{\pm\infty} + a_2^{\pm\infty} + a_3^{\pm\infty})$  are known as Kasner parameters in the literature. They satisfy  $p_1 + p_2 + p_3 = 1$  and  $(p_1)^2 + (p_2)^2 + (p_3)^2 = 1$  for both  $p_i = p_i^{\pm\infty}$ .

### 1.11 Rees algebra associated to $\mathbf{f}$

The filtration  $\mathbf{f}$ , indexed by  $\alpha \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ , has the following algebraic properties:

- $\mathbf{f}_\alpha \mathcal{E} = \bigoplus_k \mathbf{f}_\alpha \mathcal{E}^k$  where  $\mathbf{f}_\alpha \mathcal{E}^k \subseteq \mathcal{E}^k$  are linear subspaces.
- $\mathbf{f}_\alpha \mathcal{E} \subseteq \mathbf{f}_\beta \mathcal{E}$  whenever  $\alpha \leq \beta$ .
- $[\mathbf{f}_\alpha \mathcal{E}, \mathbf{f}_\beta \mathcal{E}] \subseteq \mathbf{f}_{\alpha+\beta} \mathcal{E}$ .
- $\exists \alpha : \mathbf{f}_\alpha \mathcal{E} = \mathcal{E}$ .

It has the additional property<sup>18</sup>:

- $\mathbf{f}_\alpha \mathcal{E} \cap \mathbf{f}_\beta \mathcal{E} \subseteq \sum_{\gamma \leq \alpha \text{ and } \gamma \leq \beta} \mathbf{f}_\gamma \mathcal{E}$ .

Define the Rees algebra  $\mathcal{P} = \bigoplus_\alpha s^\alpha \mathbf{f}_\alpha \mathcal{E}$  where  $s = (s', s'')$  is a pair of formal variables. One can develop things much as in §1.5, §1.6, §1.7, as discussed in §A. The upshot is that now the basic space is the graded Lie algebra

$$\begin{aligned} \mathcal{P}/s\mathcal{P} &\stackrel{\text{def}}{=} \mathcal{P}/(s'\mathcal{P} + s''\mathcal{P}) \\ &= \bigoplus_\alpha s^\alpha (\mathbf{f}_\alpha \mathcal{E} / \mathbf{f}_{<\alpha} \mathcal{E}) \end{aligned}$$

where  $\mathbf{f}_{<\alpha} \mathcal{E} = \sum_{\beta \leq \alpha \text{ and } \beta \neq \alpha} \mathbf{f}_\beta \mathcal{E}$ .

### 1.12 Reflections

Recall the 2-dim spacelike subspaces of model-space encountered in §1.9. Point-reflections within these subspaces, leaving their orthogonal complement fixed, yield graded Lie algebra automorphisms  $R', R'' \in \text{Aut}(\mathcal{E})$  with  $(R')^2 = (R'')^2 = \mathbb{1}$ . Because of the particular geometric arrangement,  $R'R'' = R''R'$ .

Hence  $\mathcal{E}$  decomposes into a direct sum of four pieces, schematically

$$\text{even}'\text{even}'' \oplus \text{even}'\text{odd}'' \oplus \text{odd}'\text{even}'' \oplus \text{odd}'\text{odd}''$$

and the bracket respects this grading. Each component of  $\mathbf{F}', \mathbf{F}''$  and  $\mathbf{f}$  is by construction invariant under both  $R', R''$ , hence also decomposes into four pieces.

---

<sup>18</sup>Explicitly  $\mathbf{f}_{p'p''} \mathcal{E} \cap \mathbf{f}_{q'q''} \mathcal{E} = \mathbf{f}_{\min(p',q') \min(p'',q'')} \mathcal{E}$ .

Having introduced these reflections, we can now also state<sup>19</sup>:

$$\begin{aligned}\mathbf{F}'_0\mathcal{E} &\subseteq (\text{even}' \text{ part of } \mathcal{E}) \\ \mathbf{F}'_1\mathcal{E} &= \mathbf{F}'_0\mathcal{E} \oplus (\text{odd}' \text{ part of } \mathcal{E}) \\ \mathbf{F}'_2\mathcal{E} &= \mathcal{E}\end{aligned}$$

Analogous for  $\mathbf{F}''$ . One can then draw various conclusions about  $\mathbf{f}$ , for example:

$$\begin{aligned}\mathbf{f}_{00}\mathcal{E} &\subseteq (\text{even'even'' part of } \mathcal{E}) \\ \mathbf{f}_{22}\mathcal{E} &= \mathcal{E}\end{aligned}$$

### 1.13 Symmetric hyperbolicity

The basic algebraic conditions on a filtration (§1.4 or §1.11) are very permissive: many uninteresting filtrations pass. In this paper we formulate additional algebraic conditions<sup>20</sup> that are specific to  $\mathcal{E}$ ; that are rather restrictive; and that we think are natural conditions. We now summarize some of their implications.

Creation operators are certain linear maps  $e : \mathcal{E}^k \rightarrow \mathcal{E}^{k+1}$  with  $e^2 = 0$ . Coming with  $\mathbf{f}$  is a set of ‘admissible’ creation operators for which

$$0 \xrightarrow{e} \mathbf{f}_\alpha\mathcal{E}^0 \xrightarrow{e} \mathbf{f}_\alpha\mathcal{E}^1 \xrightarrow{e} \mathbf{f}_\alpha\mathcal{E}^2 \xrightarrow{e} \mathbf{f}_\alpha\mathcal{E}^3 \xrightarrow{e} \mathbf{f}_\alpha\mathcal{E}^4 \xrightarrow{e} 0$$

is an exact sequence for all  $\alpha$ . For example, if  $\gamma_{(0)} \in \text{Sol}(\mathbf{f}_{00}\mathcal{E})$  and if  $f$  is a time function<sup>21</sup>, then  $e = [\gamma_{(0)}, f \cdot] - f[\gamma_{(0)}, \cdot]$  is an admissible creation operator.

The exact sequence above is a by-product of the gauge-fixing algorithm §5, a minor generalization of an algorithm in [RT]. For every choice of a gauge<sup>22</sup>  $G$  it returns a graded subspace  $(\mathbf{f}_\alpha\mathcal{E})_G \subseteq \mathbf{f}_\alpha\mathcal{E}$  that ‘splits’

$$\mathbf{f}_\alpha\mathcal{E} = (\mathbf{f}_\alpha\mathcal{E})_G \oplus e((\mathbf{f}_\alpha\mathcal{E})_G)$$

where  $e : (\mathbf{f}_\alpha\mathcal{E})_G \rightarrow \mathbf{f}_\alpha\mathcal{E}$  is injective, for all admissible creation operators  $e$ .

Let  $d_{(0)} = [\gamma_{(0)}, \cdot]$ , for all instances of this map. By design of the algorithm,  $w_{(0)} = (d_{(0)} : (\mathbf{f}_\alpha\mathcal{E})_G \rightarrow \mathbf{f}_\alpha\mathcal{E}/(\mathbf{f}_\alpha\mathcal{E})_G)$  is symmetric hyperbolic, and the complex

<sup>19</sup>This is a complete description of the filtration, except for  $\mathbf{F}'_0\mathcal{E}$ .

<sup>20</sup>In the technical jargon of this paper: Each filtration component is a  $\text{cube}_\mathcal{E}$  and has a representative that satisfies the 234-condition. See §6.1. Both  $\mathbf{F}$  and  $\mathbf{f}$  satisfy these conditions.

<sup>21</sup>A real function on the 4-dim manifold is a time function for  $\gamma_{(0)}$  if its derivative along any vector in the (image of model-space’s) future cone is positive.

<sup>22</sup>The word gauge has here a technical meaning [RT].

$d_{(0)} : \mathbf{f}_\alpha \mathcal{E} \rightarrow \mathbf{f}_\alpha \mathcal{E}$  has a deformation retraction<sup>23</sup> to  $d_{(0)} : \ker w_{(0)} \rightarrow \ker w_{(0)}$ , in particular the cohomologies are the same [RT]. The point is that, while elements of  $\mathbf{f}_\alpha \mathcal{E}$  live in four dimensions, elements of  $\ker w_{(0)}$  live in only three, since they are homogeneous solutions to a linear symmetric hyperbolic system.

We emphasize that there is no gauge-fixing going on in this paper; we only use amenability to gauge-fixing in the sense of §5 as an algebraic condition.

## 2 Language

This is a summary of parts of [RT], with an emphasis on the things we need.

Let  $\mathcal{R}$  be a commutative ring with a distinguished subring isomorphic to the reals  $\mathbb{R} \subseteq \mathcal{R}$ , such that their multiplicative units coincide. Suppose  $\text{Der}(\mathcal{R})$ <sup>24</sup> is a finite-free  $\mathcal{R}$ -module, and set  $d_{\mathcal{R}} = \text{rank}_{\mathcal{R}} \text{Der}(\mathcal{R})$ . Example:  $\mathcal{R}$  the set of real smooth functions on a manifold diffeomorphic to  $\mathbb{R}^{d_{\mathcal{R}}}$ .

Denote by  $\mathcal{C} = \mathcal{R} \oplus i\mathcal{R}$  the complexification of the ring,  $i^2 = -1$ . Let  $V$  be a finite-free  $\mathcal{C}$ -module of rank 2. Denote by  $\bar{V}$  the complex conjugate module. Let  $\mathcal{A}$  be the free  $\mathcal{C}$ -algebra generated by  $V$  and  $\bar{V}$ ,

$$\mathcal{A} = \mathcal{C} \oplus V \oplus \bar{V} \oplus VV \oplus V\bar{V} \oplus \bar{V}V \oplus \bar{V}\bar{V} \oplus VVV \oplus \dots$$

where juxtaposition means tensor product over  $\mathcal{C}$ . Let  $\mathcal{D}$  be the set of derivations on  $\mathcal{A}$  that are  $\mathbb{C}$ -linear and that preserve the grading<sup>25</sup>. Elements of  $\mathcal{D}$  need not be  $\mathcal{C}$ -linear. We sometimes write  $\mathcal{D}_{\text{vertical}} = \{\delta \in \mathcal{D} \mid \delta(\mathcal{C}) = 0\}$ .

Set  $\mathcal{L}_{\mathbb{C}} = (\bigwedge V\bar{V})\mathcal{D}$ , and again juxtaposition means tensor product over  $\mathcal{C}$ . Then  $\mathcal{L}_{\mathbb{C}}$  is a complex graded Lie algebra with bracket defined by

$$[[\omega\delta, \omega'\delta']] = (\omega \wedge \omega')[\delta, \delta'] + (\omega \wedge \delta(\omega'))\delta' - (\delta'(\omega) \wedge \omega')\delta$$

for all  $\omega, \omega' \in \bigwedge V\bar{V}$  and all  $\delta, \delta' \in \mathcal{D}$ , with  $[\delta, \delta']$  the commutator of derivations. See §B for a sample evaluation of the bracket.

<sup>23</sup>Provided the geometry associated to  $\gamma_{(0)}$  is globally hyperbolic, so that one can solve the initial value for symmetric hyperbolic systems based on  $\gamma_{(0)}$ . Then  $w_{(0)}$  is surjective.

<sup>24</sup>The space of  $\mathbb{R}$ -linear derivations on  $\mathcal{R}$

<sup>25</sup>Hence each element of  $\mathcal{D}$  maps  $\mathcal{C} \rightarrow \mathcal{C}$  and  $V \rightarrow V$  and  $\bar{V} \rightarrow \bar{V}$  and so on. Each element of  $\mathcal{D}$  is determined by its restriction to  $\mathcal{C} \oplus V \oplus \bar{V} \rightarrow \mathcal{C} \oplus V \oplus \bar{V}$ .

## 2.1 Conjugation

There is a canonical  $\mathcal{C}$ -antilinear involution on  $\mathcal{A}$ , taking  $\mathcal{C} \rightarrow \mathcal{C}$  ( $f + ig \mapsto f - ig$  for all  $f, g \in \mathcal{R}$ ) and  $V \rightarrow \bar{V}$  and  $\bar{V} \rightarrow V$  and  $V\bar{V} \rightarrow \bar{V}V$  and so on. It induces a canonical  $\mathcal{C}$ -antilinear involution  $\mathcal{D} \rightarrow \mathcal{D}$ ,  $\delta \mapsto \bar{\delta}$  by  $\bar{\delta}(\bar{a}) = \overline{\delta(a)}$  for all  $a \in \mathcal{A}$ .

By conjugation on  $V\bar{V}$  we mean conjugation followed by exchange of factors:  $V\bar{V} \rightarrow V\bar{V}$ ,  $v\bar{w} \mapsto w\bar{v}$ . We denote by  $(V\bar{V})_{\text{real}} \subseteq V\bar{V}$  the real submodule.

These conjugations induce a conjugation on  $\mathcal{L}_{\mathbb{C}}$  that we sometimes denote by  $\mathbf{C} : \mathcal{L}_{\mathbb{C}} \rightarrow \mathcal{L}_{\mathbb{C}}$ ,  $\omega\delta \mapsto \bar{\omega}\bar{\delta}$ . We denote  $\text{Re} = \frac{1}{2}(\mathbb{1} + \mathbf{C})$  and  $\text{Im} = \frac{1}{2i}(\mathbb{1} - \mathbf{C})$ .

## 2.2 The real graded Lie algebra $\mathcal{E}$

Let  $\mathcal{L} \subseteq \mathcal{L}_{\mathbb{C}}$  be the real elements, a real graded Lie algebra and an  $\mathcal{R}$ -module.

Set  $\mathcal{I}^0 = \mathcal{I}^1 = 0$ . Let  $\mathcal{I}^2 \subseteq \mathcal{L}^2$  be the elements  $x$  that satisfy  $x(\mathcal{C}) = x(V \wedge V) = 0$  and for which  $x(V)$  is contained in: the submodule of  $V\bar{V}V\bar{V}V$  that is symmetric in the three  $V$ 's. Set  $\mathcal{I}^k = (V\bar{V})_{\text{real}} \wedge \mathcal{I}^{k-1}$  recursively for  $k \geq 3$ . It is shown in [RT] that  $\mathcal{I} = \bigoplus_k \mathcal{I}^k$  is an ideal,  $[\mathcal{L}, \mathcal{I}] \subseteq \mathcal{I}$ .

Set  $\mathcal{E} = \mathcal{L}/\mathcal{I}$ , and denote the induced bracket on  $\mathcal{E}$  by  $[\cdot, \cdot]$ . Since  $V\bar{V}$  has rank 4, we have  $\mathcal{L}^k = 0$  and  $\mathcal{E}^k = 0$  whenever  $k > 4$ . The equation  $[\gamma, \gamma] = 0$  for  $\gamma \in \mathcal{E}^1$  is equivalent to the vacuum Einstein equations, if  $\mathcal{R}$  are the smooth real functions on a manifold diffeomorphic to  $\mathbb{R}^4$ , and provided the frame  $\gamma|_{\mathcal{R}}$  is nondegenerate. See [RT].

## 2.3 Vertical gauge transformations

Associated to every  $t \in \text{Aut}_{\mathcal{C}}(V)$  are automorphisms on various other spaces:

- $t \in \text{Aut}_{\mathcal{C}}(\bar{V})$  by  $t(\bar{v}) = \overline{t(v)}$  for all  $v \in V$ .
- $t \in \text{Aut}_{\mathcal{C}}(\mathcal{A})$  by  $t(ab) = t(a)t(b)$  for all  $a, b \in \mathcal{A}$ .
- $t \in \text{Aut}_{\mathcal{C}}(\mathcal{L}_{\mathbb{C}})$  by  $(t(x))(a) = t(x(t^{-1}(a)))$  for all  $x \in \mathcal{L}_{\mathbb{C}}$  and  $a \in \mathcal{A}$ .

This is a graded Lie algebra automorphism of  $\mathcal{L}_{\mathbb{C}}$  that commutes with conjugation, and therefore we have the additional graded Lie algebra automorphisms:

- $t \in \text{Aut}_{\mathcal{R}}(\mathcal{L})$  which in particular satisfies  $t(\mathcal{I}) \subseteq \mathcal{I}$ .

- $t \in \text{Aut}_{\mathcal{R}}(\mathcal{E})$ .

The so defined maps  $\text{Aut}_{\mathcal{C}}(V) \rightarrow \text{Aut}(\dots)$  are group homomorphisms.

## 2.4 Canonical conformal inner product on $(V\overline{V})_{\text{real}}$

By definition, a volume form on  $V$  is an antisymmetric nondegenerate  $\mathcal{C}$ -bilinear map  $\varepsilon : V \times V \rightarrow \mathcal{C}$ . Define a symmetric  $\mathcal{C}$ -bilinear map  $\varepsilon\bar{\varepsilon} : V\overline{V} \times V\overline{V} \rightarrow \mathcal{C}$  by

$$\varepsilon\bar{\varepsilon}(v\overline{w}, v'\overline{w'}) = \varepsilon(v, v')\overline{\varepsilon(w, w')}$$

Its restriction to an  $\mathcal{R}$ -bilinear  $(V\overline{V})_{\text{real}} \times (V\overline{V})_{\text{real}} \rightarrow \mathcal{R}$  has signature  $+\dots-$ , which is of course equivalent to one with signature  $-+++$ .

The choice of  $\varepsilon$  is unique up to multiplication by an invertible element of  $\mathcal{C}$ . Some constructions in this paper refer to a volume form  $\varepsilon$ , but this is only for convenience. Results depend on it either not at all or only in obvious ways. All objects that we define scale homogeneously in  $\varepsilon$  and  $\bar{\varepsilon}$ , but we never sum objects unless they have the same homogeneity in  $\varepsilon$  and  $\bar{\varepsilon}$ .

## 2.5 Creation operators

For every  $c \in V\overline{V}$  we define the  $\mathcal{C}$ -linear map  $e_c : \mathcal{L}_{\mathbb{C}}^k \rightarrow \mathcal{L}_{\mathbb{C}}^{k+1}$  by

$$e_c(\omega\delta) = (c \wedge \omega)\delta$$

for all  $\omega \in \bigwedge V\overline{V}$  and all  $\delta \in \mathcal{D}$ . Note that  $e_c e_{c'} + e_{c'} e_c = 0$  for all  $c, c'$ .

If  $c \in (V\overline{V})_{\text{real}}$  then we get a map  $e_c : \mathcal{L}^k \rightarrow \mathcal{L}^{k+1}$ . Since  $e_c \mathcal{I} \subseteq \mathcal{I}$  by definition of the ideal, we also get a map  $e_c : \mathcal{E}^k \rightarrow \mathcal{E}^{k+1}$ .

## 2.6 Annihilation operators

For every  $c \in V\overline{V}$  we define the  $\mathcal{C}$ -linear map  $i_c : \mathcal{L}_{\mathbb{C}}^k \rightarrow \mathcal{L}_{\mathbb{C}}^{k-1}$  by requiring that

$$i_c e_{c'} + e_{c'} i_c = \frac{1}{2} \varepsilon\bar{\varepsilon}(c, c') \mathbb{1}$$

for all  $c' \in V\overline{V}$ . This definition is recursive in  $k$ , with base case  $i_c \mathcal{L}_{\mathbb{C}}^0 = 0$ . Uniqueness of  $i_c$  is clear, but one has to check existence. The map  $c \mapsto i_c$  is  $\mathcal{C}$ -linear. Note that  $i_c$  scales homogeneously like  $\varepsilon\bar{\varepsilon}$  as a function of  $\varepsilon$ .

If  $c \in (V\overline{V})_{\text{real}}$  then we get a map  $i_c : \mathcal{L}^k \rightarrow \mathcal{L}^{k-1}$ . In general  $i_c \mathcal{I} \not\subseteq \mathcal{I}$ .



## 2.7 The anticommutation relations

For all  $c, c' \in V\overline{V}$  we have

$$\begin{aligned} \mathbf{e}_c \mathbf{e}_{c'} + \mathbf{e}_{c'} \mathbf{e}_c &= 0 \\ \mathbf{i}_c \mathbf{i}_{c'} + \mathbf{i}_{c'} \mathbf{i}_c &= 0 \\ \mathbf{i}_c \mathbf{e}_{c'} + \mathbf{e}_{c'} \mathbf{i}_c &= \frac{1}{2} \varepsilon \bar{\varepsilon}(c, c') \mathbb{1} \end{aligned}$$

which we refer to as the anticommutation relations.

## 2.8 Projection operators

For all  $c \in V\overline{V}$  such that  $\varepsilon \bar{\varepsilon}(c, c) \in \mathcal{C}$  is invertible we define

$$\mathbf{p}_c = (\frac{1}{2} \varepsilon \bar{\varepsilon}(c, c))^{-1} \mathbf{i}_c \mathbf{e}_c$$

which maps  $\mathcal{L}_{\mathbb{C}}^k \rightarrow \mathcal{L}_{\mathbb{C}}^k$  and which satisfies  $(\mathbf{p}_c)^2 = \mathbf{p}_c$ .

If  $\varepsilon \bar{\varepsilon}(c, c') = 0$  then  $\mathbf{p}_c \mathbf{p}_{c'} = \mathbf{p}_{c'} \mathbf{p}_c$ .

## Cubes

## 3 Two points of view

Recall that  $\mathcal{E} = \mathcal{L}/\mathcal{I}$ . Given a submodule  $\mathcal{L}' \subseteq \mathcal{L}$ , there are two interpretations of what the quotient of  $\mathcal{L}'$  by  $\mathcal{I}$  should be:

- $(\mathcal{L}' + \mathcal{I})/\mathcal{I}$
- $\mathcal{L}'/(\mathcal{L}' \cap \mathcal{I})$

There is of course a canonical module isomorphism between these two. We use both points of view in this paper. The first in particular when we use the bracket in  $\mathcal{E}$ . The second in particular for the gauges algorithm in §5.

### 3.1 Lemma

Suppose  $\mathcal{L}', \mathcal{L}'' \subseteq \mathcal{L}$  are submodules. Equivalent are<sup>26</sup>:

- $(\mathcal{L}' + \mathcal{I}) \cap (\mathcal{L}'' + \mathcal{I}) \subseteq (\mathcal{L}' \cap \mathcal{L}'') + \mathcal{I}$
- $(\mathcal{L}' + \mathcal{L}'') \cap \mathcal{I} \subseteq (\mathcal{L}' \cap \mathcal{I}) + (\mathcal{L}'' \cap \mathcal{I})$

and the opposite  $\supseteq$  holds trivially in both bullets.

In the way we plan to use this, the first bullet, which says that some intersection in  $\mathcal{E}$  is equivalent to an intersection in  $\mathcal{L}$ , is the one we are interested in, but the second bullet is sometimes easier to check.

## 4 Cubes $_{\mathcal{L}}$ and Cubes $_{\mathcal{E}}$

Cubes $_{\mathcal{L}}$  are submodules of  $\mathcal{L}$  that have particularly simple structure: they are representations of a set of anticommutation relations. Cubes $_{\mathcal{E}}$  are submodules of  $\mathcal{E}$  of the form  $(\text{cube}_{\mathcal{L}} + \mathcal{I})/\mathcal{I}$ . We discuss these notions in detail, since we use them a lot throughout this paper.

### 4.1 Collections of creation and annihilation operators

Let  $\mathcal{X} = (V\overline{V})_{\text{real}}$  be a submodule of rank  $0 \leq n \leq 4$  with the following property: there exists a basis of  $(V\overline{V})_{\text{real}}$ , orthogonal with respect to its canonical conformal inner product, such that  $\mathcal{X}$  is the span of the first  $n$  basis elements.

In detail, there exist  $c_1, c_2, c_3, c_4$  with

$$\begin{aligned} (V\overline{V})_{\text{real}} &= \mathcal{R}c_1 \oplus \mathcal{R}c_2 \oplus \mathcal{R}c_3 \oplus \mathcal{R}c_4 \\ \mathcal{X} &= \mathcal{R}c_1 \oplus \dots \oplus \mathcal{R}c_n \end{aligned}$$

such that

$$\varepsilon\overline{\varepsilon}(c_i, c_j) = \begin{cases} 0 & \text{if } i \neq j \\ (\text{invertible element of } \mathcal{R}) & \text{if } i = j \end{cases}$$

---

<sup>26</sup>This holds for any three subgroups of an Abelian group. We show  $\uparrow$ . Given  $x \in (\mathcal{L}' + \mathcal{I}) \cap (\mathcal{L}'' + \mathcal{I})$ , i.e.  $x = s' + i' = s'' + i''$  for some  $s' \in \mathcal{L}'$ ,  $s'' \in \mathcal{L}''$  and  $i', i'' \in \mathcal{I}$ . Then  $s' - s'' = i'' - i' \in (\mathcal{L}' + \mathcal{L}'') \cap \mathcal{I}$ . By the hypothesis,  $s' - s'' = i'' - i' = j' + j''$  with  $j' \in \mathcal{L}' \cap \mathcal{I}$  and  $j'' \in \mathcal{L}'' \cap \mathcal{I}$ . Then  $x = a + b$  with  $a = s' - j' = s'' + j''$  and  $b = i' + j' = i'' - j''$ . Since  $a \in \mathcal{L}' \cap \mathcal{L}''$  and  $b \in \mathcal{I}$  we get  $x \in (\mathcal{L}' \cap \mathcal{L}'') + \mathcal{I}$ , as required.

We often abbreviate  $e_i = e_{c_i}$ ,  $i_i = i_{c_i}$ ,  $p_i = p_{c_i}$ . The projections  $p_i$  exist and commute pairwise by §2.8. Beware that  $\mathcal{X}$  does not come with a preferred basis.

Sometimes we require  $c_1 \in (V\overline{V})_{\text{positive}} \stackrel{\text{def}}{=} \{v\overline{v} + w\overline{w} \mid v, w \text{ a basis for } V\}$  and  $n \geq 1$ ; we always say explicitly when we do.

We denote by  $e_{\mathcal{X}}$  the span of all creation operators of all elements of  $\mathcal{X}$ . Similar for  $i_{\mathcal{X}}$ . They map an element of  $\mathcal{L}$  to a subset of  $\mathcal{L}$ , or more generally a subset to a subset. Note that  $(e_{\mathcal{X}})^n \neq 0$  whereas  $(e_{\mathcal{X}})^k = 0$  for all  $k > n$ .

## 4.2 Generated submodules

Let  $\mathcal{X}$  be as in §4.1. For every graded submodule<sup>27</sup>  $\sigma \subseteq \mathcal{L}$  set

$$\langle \sigma \rangle_{\mathcal{X}} = \sigma + e_{\mathcal{X}}\sigma + \dots + (e_{\mathcal{X}})^n \sigma$$

Then  $\langle \sigma \rangle_{\mathcal{X}} \subseteq \mathcal{L}$  is itself a graded submodule. In general one cannot replace  $+$  by  $\oplus$  in this definition, but we will discuss conditions that allow one to do so.

## 4.3 Bracket estimate

Suppose  $\sigma, \sigma' \subseteq \mathcal{L}$  are graded submodules. Denote by  $W, W' \subseteq \bigwedge (V\overline{V})_{\text{real}}$  the submodules  $W = \bigwedge_{\text{tot}}(\sigma(\mathcal{X}))$  and  $W' = \bigwedge_{\text{tot}}(\sigma'(\mathcal{X}))$ . Then one has

$$\llbracket \langle \sigma \rangle_{\mathcal{X}}, \langle \sigma' \rangle_{\mathcal{X}} \rrbracket \subseteq \langle \bigwedge_{\text{tot}}(W\sigma') \rangle_{\mathcal{X}} + \langle \bigwedge_{\text{tot}}(W'\sigma) \rangle_{\mathcal{X}} + \langle \llbracket \sigma, \sigma' \rrbracket \rangle_{\mathcal{X}}$$

To see this, use the identity  $\llbracket x, e_c x' \rrbracket = \bigwedge_{\text{tot}}(x(c)x') + (-1)^{|x|} e_c \llbracket x, x' \rrbracket$  repeatedly to pull all  $e_{\mathcal{X}}$  out of the second argument of  $\llbracket \langle \sigma \rangle_{\mathcal{X}}, \langle \sigma' \rangle_{\mathcal{X}} \rrbracket$ , and then proceed similarly for the first argument.

## 4.4 $\mathcal{X}$ -oneway-cubes $_{\mathcal{L}}$ and $\mathcal{X}$ -cubes $_{\mathcal{L}}$

We say that a graded submodule  $\sigma \subseteq \mathcal{L}$  yields

- an  $\mathcal{X}$ -oneway-cube $_{\mathcal{L}}$  if and only if  $e_1 \cdots e_n : \sigma \rightarrow \mathcal{L}$  is injective<sup>28</sup>.
- an  $\mathcal{X}$ -cube $_{\mathcal{L}}$  if and only if  $i_{\mathcal{X}}\sigma = 0$ .

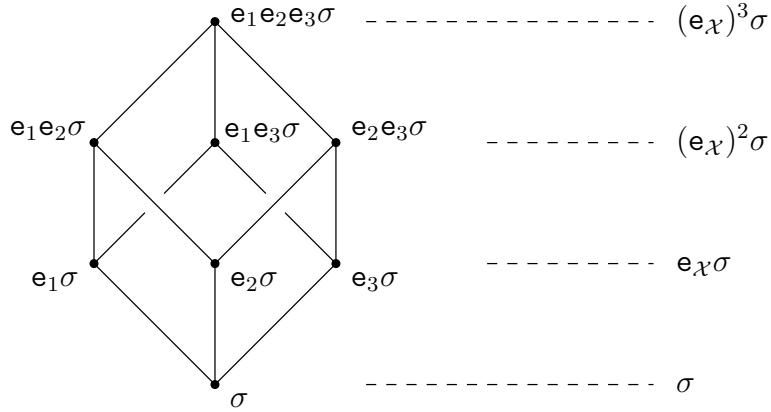
<sup>27</sup>By definition, a submodule  $S \subseteq \mathcal{L}$  is a graded submodule if and only if  $S = \bigoplus_k (S \cap \mathcal{L}^k)$ .

<sup>28</sup>This condition uses a basis  $c_1, \dots, c_n$  as in §4.1, but is basis-independent. Since  $\sigma$  is graded, this condition is equivalent to:  $e_1 \cdots e_n : \sigma \cap \mathcal{L}^k \rightarrow \mathcal{L}^{k+n}$  is injective for all  $k$ .

In the case of  $\text{oneway-cubes}_{\mathcal{L}}$ , different  $\sigma$  can yield the same  $\langle \sigma \rangle_{\mathcal{X}}$ . By contrast, for  $\text{cubes}_{\mathcal{L}}$  there is always a unique  $\sigma$  that yields a given  $\langle \sigma \rangle_{\mathcal{X}}$ , as we will see in §4.5. In this case we refer to  $\langle \sigma \rangle_{\mathcal{X}}$  itself as an  $\mathcal{X}$ -cube $_{\mathcal{L}}$ .

Every  $\mathcal{X}$ -cube $_{\mathcal{L}}$  is an  $\mathcal{X}$ -oneway-cube $_{\mathcal{L}}$ , since then  $i_1 \cdots i_n$  is a left-inverse of  $e_1 \cdots e_n : \sigma \rightarrow \mathcal{L}$ , up to multiplication by an invertible element of  $\mathcal{R}$ .

Given an  $\mathcal{X}$ -oneway-cube $_{\mathcal{L}}$ , the composition of any subset of  $e_1, \dots, e_n$  is an injective  $\sigma \rightarrow \mathcal{L}$  map, and the direct sum of the images of these  $2^n$  maps is precisely  $\langle \sigma \rangle_{\mathcal{X}}$ , a direct sum of  $2^n$  isomorphic copies of  $\sigma$ ; this follows from<sup>29</sup>  $e_i e_j + e_j e_i = 0$ . For say  $n = 3$  the picture is:



Along each edge of the cube one can move in the *upward* direction by using one of  $e_1, e_2, e_3$ . Such a move is an isomorphism between the respective bullets.

In the case of an  $\mathcal{X}$ -cube $_{\mathcal{L}}$  one can in addition move in the *downward* direction using one of  $i_1, i_2, i_3$ ; by anticommutation relations these  $i$ -moves are the inverses of the  $e$ -moves, up to multiplication by an invertible element of  $\mathcal{R}$ . Any round-trip from a bullet back to itself is also a multiple of the identity.

This discussion implies, for  $\mathcal{X}$ -oneway-cubes $_{\mathcal{L}}$  and hence for  $\mathcal{X}$ -cubes $_{\mathcal{L}}$ , the coarser but basis-independent decomposition  $\langle \sigma \rangle_{\mathcal{X}} = \sigma \oplus e_{\mathcal{X}} \sigma \oplus \dots \oplus (e_{\mathcal{X}})^n \sigma$ .

## 4.5 Equivalent characterization of $\mathcal{X}$ -cubes $_{\mathcal{L}}$

For every graded submodule  $S \subseteq \mathcal{L}$  the following are equivalent:

<sup>29</sup>One has to show, assuming say  $n = 2$ , that for all  $x, x', x'', x''' \in \sigma$  one has  $(x + e_1 x' + e_2 x'' + e_1 e_2 x''' = 0) \implies (x = x' = x'' = x''' = 0)$ . Apply  $e_1 e_2$  and use anticommutation relations to get  $e_1 e_2 x = 0$ , hence  $x = 0$  by injectivity. Then apply  $e_2$  to show  $x' = 0$  etc.

- $e_{\mathcal{X}}S \subseteq S$  and  $i_{\mathcal{X}}S \subseteq S$ .
- $S$  is an  $\mathcal{X}$ -cube $_{\mathcal{L}}$ :  $S = \langle \sigma \rangle_{\mathcal{X}}$  for a graded submodule  $\sigma \subseteq \mathcal{L}$  with  $i_{\mathcal{X}}\sigma = 0$ .

Furthermore, if either condition holds, then  $\sigma = \{x \in S \mid i_{\mathcal{X}}x = 0\}$ .

Direction  $\Uparrow$  follows from the discussion in §4.4. For  $\Downarrow$  we use a standard argument. Pick a basis as in §4.1. The hypothesis implies  $p_i S \subseteq S$ . The commuting projections  $p_1, \dots, p_n$  and  $\mathbb{1} - p_1, \dots, \mathbb{1} - p_n$  decompose  $S$  into a direct sum of  $2^n$  spaces. Denote one of these spaces by

$$\sigma = \text{image}(p_1 \cdots p_n : S \rightarrow S) = \text{image}(i_1 \cdots i_n e_1 \cdots e_n : S \rightarrow S)$$

It is graded and satisfies  $i_{\mathcal{X}}\sigma = 0$  and  $S = \langle \sigma \rangle_{\mathcal{X}}$ , and we are done with  $\Downarrow$ . It is not difficult to see that this particular choice of  $\sigma$  is forced. It is equivalent to  $\sigma = \{x \in S \mid i_{\mathcal{X}}x = 0\}$ , and also to  $\sigma = S \cap \Omega_{\mathcal{X}}$  using §4.7.

Note that  $\sigma \subseteq \mathcal{L}^0 \oplus \dots \oplus \mathcal{L}^{4-n}$ .

#### 4.6 Remark

By §4.5, the intersection of two  $\mathcal{X}$ -cubes $_{\mathcal{L}}$  is again an  $\mathcal{X}$ -cube $_{\mathcal{L}}$ , as is the span of two  $\mathcal{X}$ -cubes $_{\mathcal{L}}$ . Assuming  $i_{\mathcal{X}}\sigma = i_{\mathcal{X}}\sigma' = 0$  these operations are equivalently given by  $\langle \sigma \rangle_{\mathcal{X}} \cap \langle \sigma' \rangle_{\mathcal{X}} = \langle \sigma \cap \sigma' \rangle_{\mathcal{X}}$  and  $\langle \sigma \rangle_{\mathcal{X}} + \langle \sigma' \rangle_{\mathcal{X}} = \langle \sigma + \sigma' \rangle_{\mathcal{X}}$ .

#### 4.7 The vacuum $\Omega_{\mathcal{X}}$

We can view  $\mathcal{L}$  itself as an  $\mathcal{X}$ -cube $_{\mathcal{L}}$ ,  $\mathcal{L} = \langle \Omega_{\mathcal{X}} \rangle_{\mathcal{X}}$  with

$$\Omega_{\mathcal{X}} = \{x \in \mathcal{L} \mid i_{\mathcal{X}}x = 0\}$$

The bigger  $\mathcal{X}$  the smaller  $\Omega_{\mathcal{X}}$ . We always have  $\mathcal{L}^0 \subseteq \Omega_{\mathcal{X}} \subseteq \mathcal{L}^0 \oplus \dots \oplus \mathcal{L}^{4-n}$ .

We have a bijection

$$\begin{aligned} \{\text{graded submodules of } \Omega_{\mathcal{X}}\} &\rightarrow \{\mathcal{X}\text{-cubes}_{\mathcal{L}}\} \\ \sigma &\mapsto \langle \sigma \rangle_{\mathcal{X}} \end{aligned}$$

Conveniently,  $\Omega_{\mathcal{X}}$  is itself given by<sup>30</sup>

$$\Omega_{\mathcal{X}} = \langle \mathcal{L}^0 \rangle_{\mathcal{X}^{\perp}}$$

where by definition  $\mathcal{X}^{\perp} = \mathcal{R}_{c_{n+1}} \oplus \dots \oplus \mathcal{R}_{c_4}$ . The notation  $\mathcal{X}^{\perp}$  is justified since  $\mathcal{X}^{\perp}$  is determined by  $\mathcal{X}$ ; neither comes with a preferred basis.

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<sup>30</sup>Special case of:  $\Omega_{\mathcal{X}} = \langle \Omega_{\mathcal{X} \oplus \mathcal{Y}} \rangle_{\mathcal{Y}}$  when  $\mathcal{X} \perp \mathcal{Y}$ .

#### 4.8 The 234-condition

For every  $\mathcal{X}$ -cube $_{\mathcal{L}}$   $S$  we define

$$(\mathcal{X}, S) \text{ satisfies the 234-condition} \iff S \cap \mathcal{I} \subseteq \langle S \cap \mathcal{I}^2 \rangle_{\mathcal{X}}$$

The inclusion  $\supseteq$  is automatic, since  $S$  and  $\mathcal{I}$  are closed under  $e_{\mathcal{X}}$ . The 234-condition is necessary to run the gauges algorithm §5. Trivially  $((V\bar{V})_{\text{real}}, \mathcal{L})$  satisfies the 234-condition; it is equivalent to the definition of  $\mathcal{I}$  in terms of  $\mathcal{I}^2$ .

#### 4.9 Maximal $\mathcal{X}$ -cube $_{\mathcal{L}}$

Recall that the span and the intersection of  $\mathcal{X}$ -cubes $_{\mathcal{L}}$  are again  $\mathcal{X}$ -cubes $_{\mathcal{L}}$ .

To every graded submodule  $\mathcal{E}' \subseteq \mathcal{E}$  we can associate the biggest  $\mathcal{X}$ -cube $_{\mathcal{L}}$  for which  $(\mathcal{X}\text{-cube}_{\mathcal{L}} + \mathcal{I})/\mathcal{I} \subseteq \mathcal{E}'$ . We denote it by  $\max_{\mathcal{X}} \mathcal{E}' \subseteq \mathcal{L}$ .

Note that:

$$\max_{\mathcal{X}}(\mathcal{E}' \cap \mathcal{E}'') = \max_{\mathcal{X}} \mathcal{E}' \cap \max_{\mathcal{X}} \mathcal{E}''$$

#### 4.10 $\mathcal{X}$ -cube $_{\mathcal{E}}$

By definition, an  $\mathcal{X}$ -cube $_{\mathcal{E}}$  is a subset of  $\mathcal{E}$  of the form  $(\mathcal{X}\text{-cube}_{\mathcal{L}} + \mathcal{I})/\mathcal{I}$ .

One  $\mathcal{X}$ -cube $_{\mathcal{E}}$  can arise from more than one  $\mathcal{X}$ -cubes $_{\mathcal{L}}$ . Hence every  $\mathcal{X}$ -cube $_{\mathcal{E}}$  is an equivalence class of  $\mathcal{X}$ -cubes $_{\mathcal{L}}$ . Every  $\mathcal{X}$ -cube $_{\mathcal{E}}$  is closed under  $e_{\mathcal{X}}$ .

For every graded submodule  $\mathcal{E}' \subseteq \mathcal{E}$  we have

$$\mathcal{E}' \text{ is an } \mathcal{X}\text{-cube}_{\mathcal{E}} \iff (\max_{\mathcal{X}} \mathcal{E}' + \mathcal{I})/\mathcal{I} = \mathcal{E}'$$

The representative returned by  $\max_{\mathcal{X}}$  will be called the maximal representative.

#### 4.11 Lemma

For all  $\mathcal{X}$ -cubes $_{\mathcal{E}}$   $\mathcal{E}', \mathcal{E}'' \subseteq \mathcal{E}$  the following statements are equivalent<sup>31</sup>:

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<sup>31</sup>Clearly 3rd  $\Rightarrow$  2nd. We have 2nd  $\Rightarrow$  1st because then one of the bullets in §3.1 holds, hence the first does, hence  $(\mathcal{L}' + \mathcal{I}) \cap (\mathcal{L}'' + \mathcal{I}) = (\mathcal{L}' \cap \mathcal{L}'') + \mathcal{I}$ , and since  $\mathcal{L}' \cap \mathcal{L}''$  is an intersection of  $\mathcal{X}$ -cubes $_{\mathcal{L}}$  and hence itself an  $\mathcal{X}$ -cube $_{\mathcal{L}}$ , we are done. We have 1st  $\Rightarrow$  3rd because we can then check the first bullet in §3.1:  $((\max_{\mathcal{X}} \mathcal{E}' + \mathcal{I})/\mathcal{I}) \cap ((\max_{\mathcal{X}} \mathcal{E}'' + \mathcal{I})/\mathcal{I}) = \mathcal{E}' \cap \mathcal{E}'' = (\max_{\mathcal{X}}(\mathcal{E}' \cap \mathcal{E}'') + \mathcal{I})/\mathcal{I} = ((\max_{\mathcal{X}} \mathcal{E}' \cap \max_{\mathcal{X}} \mathcal{E}'') + \mathcal{I})/\mathcal{I}$ ; the last step uses §4.9.

- $\mathcal{E}' \cap \mathcal{E}''$  is an  $\mathcal{X}$ -cube $_{\mathcal{E}}$ .
- There exist  $\mathcal{X}$ -cubes $_{\mathcal{L}}$   $\mathcal{L}', \mathcal{L}''$  with  $(\mathcal{L}' + \mathcal{I})/\mathcal{I} = \mathcal{E}'$  and  $(\mathcal{L}'' + \mathcal{I})/\mathcal{I} = \mathcal{E}''$  such that  $\mathcal{L}', \mathcal{L}''$  satisfy one of the bullets in §3.1.
- $\mathcal{L}' = \max_{\mathcal{X}} \mathcal{E}'$  and  $\mathcal{L}'' = \max_{\mathcal{X}} \mathcal{E}''$  satisfy one of the bullets in §3.1.

Furthermore, if the 2nd bullet holds then  $\mathcal{E}' \cap \mathcal{E}''$  has representative  $\mathcal{L}' \cap \mathcal{L}''$ , and if the 3rd bullet holds then  $\mathcal{E}' \cap \mathcal{E}''$  has maximal representative  $\max_{\mathcal{X}} \mathcal{E}' \cap \max_{\mathcal{X}} \mathcal{E}''$ .

## 5 Gauges algorithm in a cube $_{\mathcal{L}}$

This §5 is not used in the rest of this paper. However it motivates our working with cubes $_{\mathcal{E}}$  and cubes $_{\mathcal{L}}$  throughout this paper, as well as the 234-condition.

This §5 must be read together with the gauges algorithm in [RT]. It concerns a straightforward generalization of the algorithm in [RT], that instead of  $\mathcal{L}$  works entirely within a cube $_{\mathcal{L}}$ , provided this cube $_{\mathcal{L}}$  satisfies conditions that we state below, including the 234-condition.

To see how this §5 is applied, we refer to [RT].

### 5.1 Input

The algorithm takes as input a positive definite Hermitian form  $G$  as in the original algorithm in [RT], satisfying conditions (i1), (i2), (i3) stated there.

It additionally takes as input:

- An  $\mathcal{X} \subseteq (V\overline{V})_{\text{real}}$  that satisfies §4.1 with  $n \geq 1$  and  $c_1 \in (V\overline{V})_{\text{positive}}$ .
- A graded submodule  $\mathbf{S}\mathcal{L} \subseteq \mathcal{L}$  such that:
  - $\mathbf{S}\mathcal{L}$  is an  $\mathcal{X}$ -cube $_{\mathcal{L}}$ , see §4.5.
  - The pair  $(\mathcal{X}, \mathbf{S}\mathcal{L})$  satisfies the 234-condition, see §4.8.

Set  $\mathbf{S}\mathcal{I} = \mathcal{I} \cap \mathbf{S}\mathcal{L}$  and  $\mathbf{S}\mathcal{E} = \mathbf{S}\mathcal{L}/\mathbf{S}\mathcal{I}$ . We additionally require:

- $\mathbf{S}\mathcal{L}^k$  is finite-free and has finite-free complement in  $\mathcal{L}^k$ .  
 $\mathbf{S}\mathcal{I}^k$  is finite-free and has finite-free complement in  $\mathbf{S}\mathcal{L}^k$ .  
 $\mathbf{S}\mathcal{E}^{k+1}/\mathbf{e}_1(\mathbf{S}\mathcal{E}^k)$  is finite-free, where  $\mathbf{e}_1 = \mathbf{e}_{c_1}$ .

Observe that the conditions that  $\mathbf{G}$  must satisfy are completely separate from the conditions that  $\mathcal{X}$  and  $\mathbf{SL}$  must satisfy.

## 5.2 Output

The output is very similar to [RT], but here everything refers to  $\mathbf{SE}$  and  $\mathcal{X}$ .

Set  $\mathbf{SE}_G^{-1} = 0$ .

The algorithm returns sequentially for  $k = -1, 0, 1, 2, \dots$  an  $\mathcal{R}$ -bilinear map

$$\mathbf{Sb}^k : \mathbf{SE}_G^k \times \mathbf{SE}^{k+1} \rightarrow \mathcal{R}$$

such that we have:

**S(o1)<sub>k</sub>**  $\mathbf{SE}_G^k \times \mathbf{SE}_G^k \rightarrow \mathcal{R}$ ,  $(x, y) \mapsto \mathbf{Sb}^k(x, e_c y)$  is symmetric for all  $c \in \mathcal{X}$ , and positive definite for all  $c \in (V\bar{V})_{\text{positive}} \cap \mathcal{X}$ .

**S(o2)<sub>k</sub>** Define  $\mathbf{SE}_G^{k+1} \subseteq \mathbf{SE}^{k+1}$  by

$$\mathbf{Sb}^k(\mathbf{SE}_G^k, \mathbf{SE}_G^{k+1}) = 0$$

Then for all  $c \in (V\bar{V})_{\text{positive}} \cap \mathcal{X}$ ,

$$\mathbf{SE}^{k+1} = e_c(\mathbf{SE}_G^k) \oplus \mathbf{SE}_G^{k+1}$$

is an internal direct sum decomposition, and  $e_c : \mathbf{SE}_G^k \rightarrow \mathbf{SE}^{k+1}$  is injective.

**S(o3)<sub>k</sub>**  $\mathbf{SE}_G^{k+1}$  is finite-free.

## 5.3 The generalized algorithm (outline)

The algorithm is completely analogous to the one in [RT], except that everything takes place in the  $\mathcal{X}$ -cube <sub>$\mathcal{L}$</sub>   $\mathbf{SL}$ . One always restricts to  $c \in \mathcal{X}$  so that one stays in the cube <sub>$\mathcal{L}$</sub>  at all intermediate steps. The 234-condition is used in key places to show that things descend from  $\mathbf{SL}$  to  $\mathbf{SE}$ .

Note that **S(o1)<sub>k</sub>** implies **S(o2)<sub>k</sub>**, and **S(o2)<sub>k-1</sub>** and **S(o2)<sub>k</sub>** imply **S(o3)<sub>k</sub>**. Hence one only has to show **S(o1)<sub>k</sub>** at each step.

The map  $k$  in [RT] satisfies  $k(\mathbf{SI}^3) \subseteq \mathbf{SL}^1$  by the 234-condition and since we have an  $\mathcal{X}$ -cube <sub>$\mathcal{L}$</sub> . Similar to [RT] we denote by  $\mathbf{SL}_G^k \subseteq \mathbf{SL}^k$  the preimage



of  $\mathbf{SE}_g^k \subseteq \mathbf{SE}^k$  under  $\mathbf{SL}^k \rightarrow \mathbf{SE}^k$ . Therefore  $\mathbf{SI}^k \subseteq \mathbf{SL}_g^k$  and  $\mathbf{SE}_g^k = \mathbf{SL}_g^k / \mathbf{SI}^k$ . Set  $\mathbf{SE}_g^{-1} = 0$  and  $\mathbf{Sb}^{-1} = 0$ .

Set  $\mathbf{Sb}^0 = \mathbf{B}^0$  on  $\mathbf{SL}_g^0 \times \mathbf{SL}^1$ . It descends to  $\mathbf{SE}_g^0 \times \mathbf{SE}^1$  and satisfies  $\mathbf{S(o1)}_0$ .

Set  $\mathbf{Sb}^1 = \mathbf{B}^1$  on  $\mathbf{SL}_g^1 \times \mathbf{SL}^2$ . It descends to  $\mathbf{SE}_g^1 \times \mathbf{SE}^2$  and satisfies  $\mathbf{S(o1)}_1$ .

Define  $\mathbf{Sa}^2 : \mathbf{SL}_g^2 \rightarrow \mathbf{SL}^0$  by  $\mathbf{B}^1(\mathbf{SL}^1, (\mathbf{j Sa}^2 + \mathbb{1})(\mathbf{SL}_g^2)) = 0$ . To show the existence of a unique such map, one uses the fact that  $(V\bar{V})_{\text{positive}} \cap \mathcal{X}$  is nonempty. Set  $\mathbf{Sa}^2 = \mathbf{j Sa}^2 + \mathbb{1} : \mathbf{SL}_g^2 \rightarrow (V\bar{V})^{\otimes 2} \mathcal{D}$ . Then  $\mathbf{Sa}^2 = \mathbb{1}$  on  $\mathbf{SI}^2$ . Set  $\mathbf{Sb}^2 = \mathbf{B}^2(\mathbf{Sa}^2(\cdot), \cdot)$  on  $\mathbf{SL}_g^2 \times \mathbf{SL}^3$ . For all  $x, y \in \mathbf{SL}_g^2$  and  $c \in \mathcal{X}$  we have  $\mathbf{Sb}^2(x, \mathbf{e}_c y) = \mathbf{B}^2(\mathbf{Sa}^2(x), c\mathbf{Sa}^2(y))$ ; one instance where having a  $\text{cube}_{\mathcal{L}}$  is essential. It descends to  $\mathbf{SE}_g^2 \times \mathbf{SE}^3$ ; one instance where the 234-condition is essential. It satisfies  $\mathbf{S(o1)}_2$ .

Define  $\mathbf{Sa}^3 : \mathbf{SL}_g^3 \rightarrow \mathbf{SL}^1$  by  $\mathbf{B}^2(\mathbf{j SL}^0 \oplus \mathbf{SL}^2, (\mathbf{j Sa}^3 + \mathbb{1})(\mathbf{SL}_g^3)) = 0$ . Set  $\mathbf{Sa}^3 = \mathbf{j Sa}^3 + \mathbb{1} : \mathbf{SL}_g^3 \rightarrow (V\bar{V})^{\otimes 3} \mathcal{D}$ . Then  $\mathbf{Sa}^3 = \mathbf{jk} + \mathbb{1}$  on  $\mathbf{SI}^3$ ; one of several instances where  $\mathbf{k}(\mathbf{SI}^3) \subseteq \mathbf{SL}^1$  is used. Set  $\mathbf{Sb}^3 = \mathbf{B}^3(\mathbf{Sa}^3(\cdot), \cdot)$  on  $\mathbf{SL}_g^3 \times \mathbf{SL}^4$ . For all  $x, y \in \mathbf{SL}_g^3$  and  $c \in \mathcal{X}$  we have  $\mathbf{Sb}^3(x, \mathbf{e}_c y) = \mathbf{B}^3(\mathbf{Sa}^3(x), c\mathbf{Sa}^3(y))$ . It descends to  $\mathbf{SE}_g^3 \times \mathbf{SE}^4$ . It satisfies  $\mathbf{S(o1)}_3$ .

We show that  $\mathbf{SE}_g^4 = 0$ . Pick a  $c \in (V\bar{V})_{\text{positive}} \cap \mathcal{X}$ . We have<sup>32</sup>  $\mathbf{e}_c(\mathbf{SL}^3) = \mathbf{SL}^4$ . Hence  $\mathbf{e}_c(\mathbf{SE}^3) = \mathbf{SE}^4$ . But we also have  $\mathbf{e}_c(\mathbf{SE}^3) = \mathbf{e}_c(\mathbf{SE}_g^3)$  by  $\mathbf{S(o2)}_2$ . Hence  $\mathbf{SE}^4 = \mathbf{e}_c(\mathbf{SE}_g^3)$ , and now  $\mathbf{S(o2)}_3$  implies  $\mathbf{SE}_g^4 = 0$ .

## 6 Filtration by $\text{cubes}_{\mathcal{E}}$

In this section we use the letter  $\mathbf{f}$  to denote an unspecified filtration. The same letter is going to stand for a particular filtration, later in this paper.

### 6.1 Definition of a filtration by $\mathcal{X}$ - $\text{cubes}_{\mathcal{E}}$

Suppose

- $\mathcal{X}$  satisfies §4.1 with  $n \geq 1$  and  $c_1 \in (V\bar{V})_{\text{positive}}$ .

We say that a collection  $(\mathbf{f}_{\alpha} \mathcal{E})_{\alpha}$ , where the index  $\alpha$  runs over a power of  $\mathbb{Z}_{\geq 0}$ , is a filtration by  $\mathcal{X}$ - $\text{cubes}_{\mathcal{E}}$  if and only if the following conditions hold:

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<sup>32</sup>By  $\mathbf{e}_c(\mathbf{SL}^4) = 0$  and anticommutation relations and having a  $\text{cube}_{\mathcal{L}}$ .

- $\mathbf{f}_\alpha \mathcal{E} \subseteq \mathcal{E}$  is a graded submodule.
- $\mathbf{f}_\alpha \mathcal{E} \subseteq \mathbf{f}_\beta \mathcal{E}$  if  $\alpha \leq \beta$ .
- $[\mathbf{f}_\alpha \mathcal{E}, \mathbf{f}_\beta \mathcal{E}] \subseteq \mathbf{f}_{\alpha+\beta} \mathcal{E}$ .
- $\exists \alpha : \mathbf{f}_\alpha \mathcal{E} = \mathcal{E}$ .
- $\mathbf{f}_\alpha \mathcal{E} \cap \mathbf{f}_\beta \mathcal{E} \subseteq \sum_{\gamma \leq \alpha \text{ and } \gamma \leq \beta} \mathbf{f}_\gamma \mathcal{E}$ .
- $\mathbf{f}_\alpha \mathcal{E}$  is an  $\mathcal{X}$ -cube $_{\mathcal{E}}$ .
- $\mathbf{f}_\alpha \mathcal{E}$  has a representative that satisfies, with  $\mathcal{X}$ , the 234-condition<sup>33</sup>.
- $\gamma(\mathcal{R}) \subseteq \mathcal{X}$  for all  $\gamma \in \mathbf{f}_0 \mathcal{E}^1$ .

The last three are motivated by §5.

## 6.2 Remarks: Constructing such a filtration

Here is a rough strategy for constructing a filtration as in §6.1. There is of course no guarantee that this strategy works, it is just a strategy.

Choose  $\mathcal{X}$ . Calculate  $\Omega_{\mathcal{X}} = \langle \mathcal{L}^0 \rangle_{\mathcal{X}^\perp}$ .

Choose  $\mathbf{f}_\alpha \Omega_{\mathcal{X}} \subseteq \Omega_{\mathcal{X}}$  and set  $\mathbf{f}_\alpha \mathcal{L} = \langle \mathbf{f}_\alpha \Omega_{\mathcal{X}} \rangle_{\mathcal{X}}$ .

Check that  $[\mathbf{f}_\alpha \mathcal{L}, \mathbf{f}_\beta \mathcal{L}] \subseteq \mathbf{f}_{\alpha+\beta} \mathcal{L} + \mathcal{I}$ ; one can use the bracket estimate §4.3.

Check that  $(\mathcal{X}, \mathbf{f}_\alpha \mathcal{L})$  satisfies the 234-condition.

Check that  $\mathbf{f}_\alpha \mathcal{E} = (\mathbf{f}_\alpha \mathcal{L} + \mathcal{I})/\mathcal{I}$  satisfies all conditions in §6.1.

## 6.3 Remarks: Intersecting two such filtrations

Suppose  $\mathbf{F}', \mathbf{F}''$  are two filtrations as in §6.1 relative to the same  $\mathcal{X}$ . Set

$$\mathbf{f}_{\alpha' \alpha''} \mathcal{E} = \mathbf{F}'_{\alpha'} \mathcal{E} \cap \mathbf{F}''_{\alpha''} \mathcal{E}$$

where  $\alpha' \alpha''$  is a shorthand for a pair of indices<sup>34</sup>. Then  $\mathbf{f}$  automatically satisfies many, but not necessarily all, properties in §6.1 relative to  $\mathcal{X}$ .

In particular,  $\mathbf{f}_{\alpha' \alpha''} \mathcal{E}$  is not obviously an  $\mathcal{X}$ -cube $_{\mathcal{E}}$ . The intersection of a representative of  $\mathbf{F}'_{\alpha'} \mathcal{E}$  with a representative of  $\mathbf{F}''_{\alpha''} \mathcal{E}$  is not obviously a representative of  $\mathbf{f}_{\alpha' \alpha''} \mathcal{E}$ . See §4.11 for details.

<sup>33</sup>One may want to consider a choice of such representatives as being part of the filtration.

<sup>34</sup> $(\alpha' \alpha'' \leq \beta' \beta'') \iff (\alpha' \leq \beta' \text{ and } \alpha'' \leq \beta'') \text{ and } \alpha' \alpha'' + \beta' \beta'' = (\alpha' + \beta')(\alpha'' + \beta'')$

## Structure associated to a decomposition of $V$

Suppose a direct sum decomposition of  $V$  is given,

$$V = V_- \oplus V_+ \quad (1)$$

where both summands are finite-free  $\mathcal{C}$ -modules of rank 1.

In §9 we construct a filtration by cubes $_{\mathcal{E}}$  associated to (1). The sections leading up to §9 are to get acquainted with (1).

## 7 Associated decomposition of $\mathcal{L}$

Given is a decomposition  $V = V_- \oplus V_+$ . See (1).

### 7.1 Notation

We introduce gradings on various  $\mathcal{C}$ -modules, using the notation

$$(\text{module}) = \bigoplus_{i,j \in \frac{1}{2}\mathbb{Z}} \mathbf{G}_{i,j}(\text{module})$$

In all figures below,  $i$  is constant on vertical lines and increases towards the right in steps of  $\frac{1}{2}$ , whereas the index  $j$  is constant on horizontal lines and increases towards the top. No bullet means that the corresponding component vanishes. The presence of a bullet means that the corresponding component has rank  $\geq 1$ .

We also define coarser gradings: for all  $\ell \in \mathbb{Z}$  we set  $\mathbf{G}_{\ell} = \bigoplus_{i-j=\ell} \mathbf{G}_{i,j}$  and we set  $\mathbf{G}_{\text{even (odd)}} = \bigoplus_{\ell \text{ even (odd)}} \mathbf{G}_{\ell}$ .

Whenever defined, conjugation  $\mathbf{C}$  maps  $\mathbf{G}_{i,j} \rightarrow \mathbf{G}_{j,i}$ .

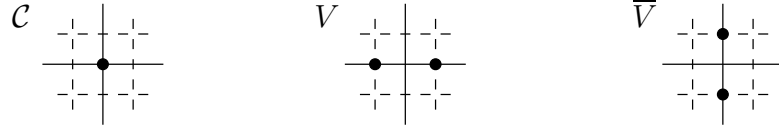
The grading on a tensor product is always the tensor product grading.

### 7.2 Grading on $\mathcal{C}$ and $V$ and $\bar{V}$

Set

$$\begin{array}{lll} \mathbf{G}_{0,0}\mathcal{C} = \mathcal{C} & \mathbf{G}_{-\frac{1}{2},0}V = V_- & \mathbf{G}_{0,-\frac{1}{2}}\bar{V} = \bar{V}_- \\ & \mathbf{G}_{\frac{1}{2},0}V = V_+ & \mathbf{G}_{0,\frac{1}{2}}\bar{V} = \bar{V}_+ \end{array}$$

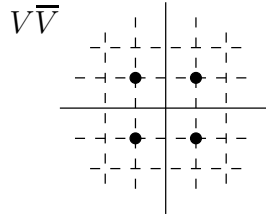
and set all other components to zero:



### 7.3 Grading on the free algebra $\mathcal{A}$

The grading on  $\mathcal{C} \oplus V \oplus \bar{V}$  induces a grading on  $\mathcal{A}$ .

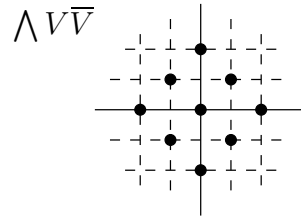
For example,  $V\bar{V} \subseteq \mathcal{A}$  has nontrivial components:



For example, the lower right bullet is  $\mathbf{G}_{\frac{1}{2}, -\frac{1}{2}} V\bar{V} = V_+ \bar{V}_-$ .

### 7.4 Grading on $\wedge V\bar{V}$

Nontrivial components:



For example, the center bullet and the rightmost bullet are given by

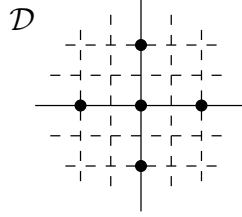
$$\begin{aligned} \mathbf{G}_{0,0}(\wedge V\bar{V}) &= \mathcal{C} \oplus (V_+ \bar{V}_+ \wedge V_- \bar{V}_-) \oplus (V_+ \bar{V}_- \wedge V_- \bar{V}_+) \oplus (\wedge^4 V\bar{V}) \\ \mathbf{G}_{1,0}(\wedge V\bar{V}) &= V_+ \bar{V}_- \wedge V_+ \bar{V}_+ \end{aligned}$$

## 7.5 Grading on $\mathcal{D}$

Define

$$\mathbf{G}_{i,j}\mathcal{D} = \{\delta \in \mathcal{D} \mid \forall i', j' : \delta(\mathbf{G}_{i',j'}\mathcal{A}) \subseteq \mathbf{G}_{i+i',j+j'}\mathcal{A}\}$$

Then  $\mathcal{D} = \bigoplus_{i,j \in \frac{1}{2}\mathbb{Z}} \mathbf{G}_{i,j}\mathcal{D}$ . Nontrivial components:

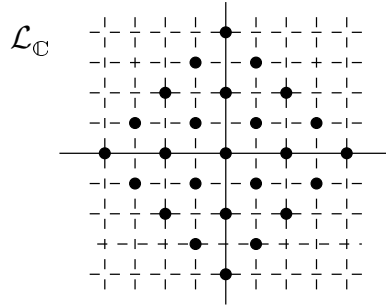


The four outer bullets are all in  $\mathcal{D}_{\text{vertical}}$ .

Center bullet:  $\delta \in \mathbf{G}_{0,0}\mathcal{D}$  if and only if  $\delta(\mathcal{C}) \subseteq \mathcal{C}$ ,  $\delta(V_{\pm}) \subseteq V_{\pm}$ ,  $\delta(\overline{V_{\pm}}) \subseteq \overline{V_{\pm}}$ , hence  $\text{rank}_{\mathcal{C}} \mathbf{G}_{0,0}\mathcal{D} = 4 + d_{\mathcal{R}}$ . Rightmost bullet:  $\delta \in \mathbf{G}_{1,0}\mathcal{D}$  if and only if  $\delta(\mathcal{C}) = \delta(V_+) = \delta(\overline{V_{\pm}}) = 0$  and  $\delta(V_-) \subseteq V_+$ , hence  $\text{rank}_{\mathcal{C}} \mathbf{G}_{1,0}\mathcal{D} = 1$ .

## 7.6 Grading on the complex graded Lie algebra $\mathcal{L}_{\mathbb{C}}$

By combining §7.4 and §7.5 we get:



One actually has a 3-grading,

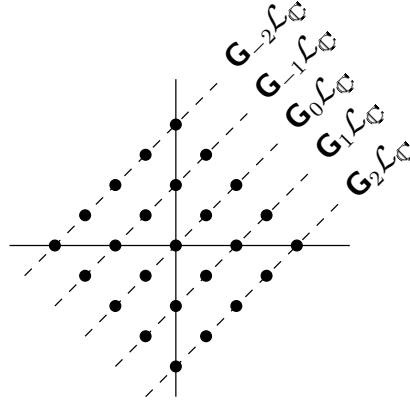
$$\mathcal{L}_{\mathbb{C}} = \bigoplus_k \bigoplus_{i,j \in \frac{1}{2}\mathbb{Z}} \mathbf{G}_{i,j}\mathcal{L}_{\mathbb{C}}^k$$

Hence the figure is really an overlay of five figures, one for each of  $\mathcal{L}_{\mathbb{C}}^0, \dots, \mathcal{L}_{\mathbb{C}}^4$ . Each of the four corner bullets has rank 1 and is contained in  $\mathcal{L}_{\mathbb{C}}^2$ . Each of the eight corner-neighbors has rank 2: one from  $\mathcal{L}_{\mathbb{C}}^1$ , one from  $\mathcal{L}_{\mathbb{C}}^3$ .

By construction, the grading respects the bracket:

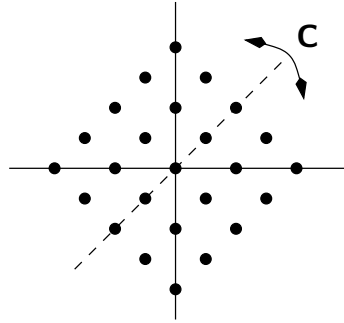
$$[\mathbf{G}_{i,j}\mathcal{L}_{\mathbb{C}}, \mathbf{G}_{i',j'}\mathcal{L}_{\mathbb{C}}] \subseteq \mathbf{G}_{i+i',j+j'}\mathcal{L}_{\mathbb{C}}$$

The coarser grading



also respects the bracket,  $[\mathbf{G}_{\ell}\mathcal{L}_{\mathbb{C}}, \mathbf{G}_{\ell'}\mathcal{L}_{\mathbb{C}}] \subseteq \mathbf{G}_{\ell+\ell'}\mathcal{L}_{\mathbb{C}}$ .

Conjugation, defined in §2.1, reflects about the diagonal:

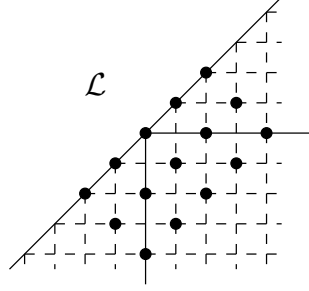


## 7.7 Grading on $\mathcal{L}$ that respects the bracket only in a restricted sense

Recall that  $\mathcal{L} \subseteq \mathcal{L}_{\mathbb{C}}$  is the subset of real elements. Since this is an  $\mathcal{R}$ -module, the notation here does not quite agree with §7.1. Define

$$\mathbf{G}_{i,j}\mathcal{L} = \{\text{real elements of } \mathbf{G}_{i,j}\mathcal{L}_{\mathbb{C}} + \mathbf{G}_{j,i}\mathcal{L}_{\mathbb{C}}\}$$

Then  $\mathbf{G}_{i,j}\mathcal{L} = \mathbf{G}_{j,i}\mathcal{L}$  and  $\mathcal{L} = \bigoplus_{i \geq j} \mathbf{G}_{i,j}\mathcal{L}$ . Hence we restrict figures to  $i \geq j$ :



This grading respects the bracket only in a restricted sense:

$$[\mathbf{G}_{i,j}\mathcal{L}, \mathbf{G}_{i',j'}\mathcal{L}] \subseteq \mathbf{G}_{i+i',j+j'}\mathcal{L} + \mathbf{G}_{i+j',j+i'}\mathcal{L}$$

In particular the bracket respects  $\mathbf{G}\mathcal{L} = \mathbf{G}_{\text{even}}\mathcal{L} \oplus \mathbf{G}_{\text{odd}}\mathcal{L}$  as a  $\mathbb{Z}_2$ -grading.

## 8 Associated decomposition of $\mathcal{I}$

Given is a decomposition  $V = V_- \oplus V_+$ . See (1).

In this section we use bases  $V_{\pm} = \mathcal{C}v_{\pm}$ . This is a matter of convenience; final results are  $v_{\pm}$ -independent, as we repeatedly point out.

### 8.1 Creation and annihilation operators $e_{\pm}$ and $i_{\pm}$

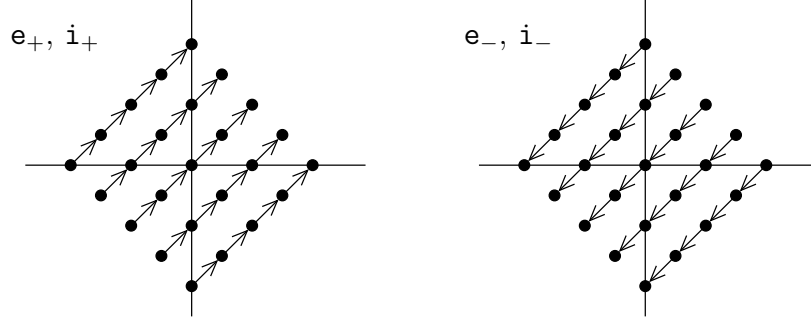
Abbreviate

$$\begin{aligned} e_+ &= e_{v_+ \overline{v_+}} & i_+ &= i_{v_+ \overline{v_+}} \\ e_- &= e_{v_- \overline{v_-}} & i_- &= i_{v_- \overline{v_-}} \end{aligned}$$

We use these operators in a way that does not depend on the choice of  $v_{\pm}$ , nor on the choice of the auxiliary volume form hidden in  $i_{\pm}$ . Note that

$$\begin{aligned} e_+ i_+ + i_+ e_+ &= e_- i_- + i_- e_- = 0 \\ e_+ i_- + i_- e_+ &= e_- i_+ + i_+ e_- = (\text{an invertible element of } \mathcal{C}) \mathbb{1} \end{aligned}$$

In addition to these, the anticommutator of any two creation operators vanishes, as does the anticommutator of any two annihilation operators. The creation and annihilation operators shift the  $\mathbf{G}$ -grading on  $\mathcal{L}_{\mathbb{C}}$  as follows:



These figures do not show  $e_{\pm}(\mathcal{L}_{\mathbb{C}}^k) \subseteq \mathcal{L}_{\mathbb{C}}^{k+1}$  and  $i_{\pm}(\mathcal{L}_{\mathbb{C}}^k) \subseteq \mathcal{L}_{\mathbb{C}}^{k-1}$ .

## 8.2 Decomposition of the ideal $\mathcal{I}_{\mathbb{C}}$

Set  $\mathcal{I}_{\mathbb{C}} = \mathcal{I} \oplus i\mathcal{I}$ , a  $\mathcal{C}$ -submodule and ideal of the graded Lie algebra  $\mathcal{L}_{\mathbb{C}}$ . It has an alternative decomposition into  $\mathcal{C}$ -modules,  $\mathcal{I}_{\mathbb{C}} = \mathcal{I}_* \oplus \mathcal{C}\mathcal{I}_*$  with

$$\mathcal{I}_* = \{x \in \mathcal{I}_{\mathbb{C}} \mid x(\bar{V}) = 0\}$$

In this §8.2 we decompose  $\mathcal{I}_*$ . This decomposition immediately induces a decomposition of  $\mathcal{I}$ , via the  $\mathcal{R}$ -module isomorphism  $\text{Re} : \mathcal{I}_* \rightarrow \mathcal{I}$ .

For each  $z \in \mathbb{C}$  abbreviate  $v_z = v_- + zv_+$ .

Define  $\xi_z \in \mathcal{L}_{\mathbb{C}}^2$  by  $\xi_z(\mathcal{C}) = \xi_z(\bar{V}) = 0$  and

$$\xi_z(v_w) = (w - z)(v_z \bar{v}_+ \wedge v_z \bar{v}_-)v_z$$

To extract  $\xi_z(v_{\pm})$  from this definition, expand both sides as linear polynomials in  $w$  and compare coefficients. It is immediate that<sup>35</sup>  $\xi_z \in \mathcal{I}_*^2$ .

Since  $\xi_z$  is polynomial in  $z$  of degree four, we can assign names to its coefficients<sup>36</sup>,  $\xi_z = \xi_{-2} + z\xi_{-1} + z^2\xi_0 + z^3\xi_1 + z^4\xi_2$ . The rank 1 modules  $\Xi_j = \mathcal{C}\xi_j$  are  $v_{\pm}$ -independent<sup>37</sup> and we have

$$\mathcal{I}_*^2 = \Xi_{-2} \oplus \Xi_{-1} \oplus \Xi_0 \oplus \Xi_1 \oplus \Xi_2$$

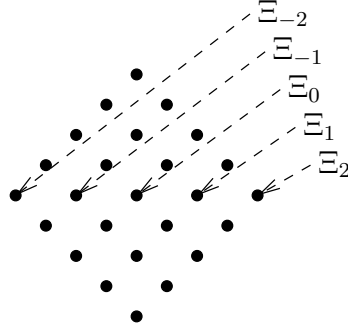
with  $\text{rank}_{\mathcal{C}} \mathcal{I}_*^2 = 5$ . We have:

<sup>35</sup>By definition  $\xi_z(\mathcal{C}) = \xi_z(\bar{V}) = 0$ , and  $\xi_z(V)$  is contained in the submodule of  $V\bar{V}V\bar{V}V$  symmetric in the three  $V$ 's. To check  $\xi_z(V \wedge V) = 0$ , note that  $\xi_z(v_w v_{w'}) = (w + w' - 2z)(v_z \bar{v}_+ \wedge v_z \bar{v}_-)v_z v_z + (w - z)(w' - z)(v_z \bar{v}_+ \wedge v_z \bar{v}_-)(v_z v_+ + v_+ v_z)$  is symmetric in  $w, w'$ .

<sup>36</sup>We are abusing notation here, e.g.  $\xi_{-2}$  is not equal to  $\xi_z$  evaluated at  $z = -2$ .

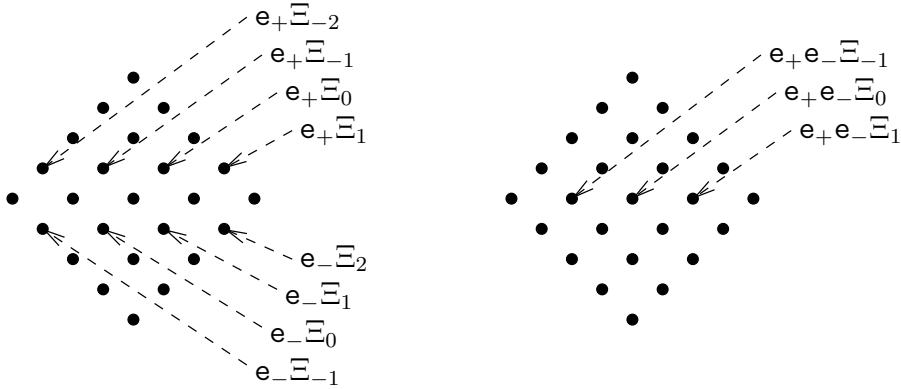
<sup>37</sup>By contrast,  $\mathcal{R}\xi_j$  does depend on  $v_{\pm}$ . The one exception is  $j = 0$ , in fact  $\Xi_0 = \mathcal{R}\xi_0 \oplus i\mathcal{R}\xi_0$  is a  $v_{\pm}$ -independent decomposition into two modules of  $\mathcal{R}$ -rank 1.





where an arrow means ‘contained in’. Note that the  $\mathbf{C}\Xi_i$  are on the vertical axis; that  $\Xi_0 \cap \mathbf{C}\Xi_0 = 0$ ; and that  $\Xi_{\pm 2}$  and  $\mathbf{C}\Xi_{\pm 2}$  are equal to the four corner bullets.

Note that  $e_{\pm}\Xi_i \subseteq \mathcal{I}_*^3$  and  $e_+e_-\Xi_i \subseteq \mathcal{I}_*^4$ . We have



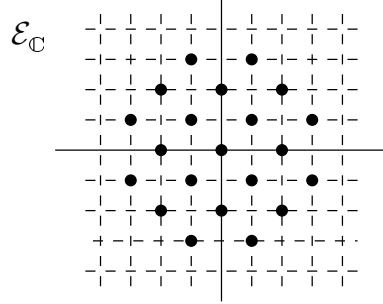
where an arrow means ‘contained in’. Note that  $e_-\Xi_{-2} = e_+\Xi_2 = 0$ . We have

$$\begin{aligned}\mathcal{I}_*^3 &= e_-\Xi_{-1} \oplus e_-\Xi_0 \oplus e_-\Xi_1 \oplus e_-\Xi_2 \oplus e_+\Xi_{-2} \oplus e_+\Xi_{-1} \oplus e_+\Xi_0 \oplus e_+\Xi_1 \\ \mathcal{I}_*^4 &= e_+e_-\Xi_{-1} \oplus e_+e_-\Xi_0 \oplus e_+e_-\Xi_1\end{aligned}$$

where each summand has rank 1, hence  $\text{rank}_{\mathcal{C}} \mathcal{I}_*^3 = 8$  and  $\text{rank}_{\mathcal{C}} \mathcal{I}_*^4 = 3$ .

### 8.3 Grading on the complex graded Lie algebra $\mathcal{E}_{\mathbb{C}}$

Set  $\mathcal{E}_{\mathbb{C}} = \mathcal{L}_{\mathbb{C}}/\mathcal{I}_{\mathbb{C}}$ . The nontrivial components of the  $\mathbf{G}$ -grading on  $\mathcal{E}_{\mathbb{C}}$  are



Each of the eight corner bullets has rank 1 and is contained in  $\mathcal{E}_\mathbb{C}^1$ .

## 9 Associated filtration by cubes $_{\mathcal{E}}$

Given is a decomposition  $V = V_- \oplus V_+$ . See (1).

We construct a filtration as in §6.1, indexed by  $\mathbb{Z}_{\geq 0}$ . We loosely follow §6.2.

### 9.1 Invariance condition

We require that the filtration that we are about to construct be invariant, in the sense of §2.3, under the subgroup

$$\{t \in \text{Aut}_c(V) \mid t(V_{\pm}) \subseteq V_{\pm} \text{ or } t(V_{\pm}) \subseteq V_{\mp}\}$$

### 9.2 Choice of $\mathcal{X}$

Set<sup>38</sup>

$$\begin{aligned} \mathcal{X} &= (V_- \overline{V_-} \oplus V_+ \overline{V_+})_{\text{real}} \\ &= \mathcal{R}c_0 \oplus \mathcal{R}c_3 \end{aligned}$$

where an explicit orthogonal basis of  $(V\overline{V})_{\text{real}}$  as in §4.1 is given by

$$\begin{aligned} c_0 &= v_+ \overline{v_+} + v_- \overline{v_-} & c_1 &= v_+ \overline{v_-} + v_- \overline{v_+} \\ c_3 &= v_+ \overline{v_+} - v_- \overline{v_-} & c_2 &= i(v_+ \overline{v_-} - v_- \overline{v_+}) \end{aligned}$$

Note that  $c_0 \in (V\overline{V})_{\text{positive}}$ .

Then  $\mathbf{e}_{\mathcal{X}} = \mathcal{R}\mathbf{e}_- + \mathcal{R}\mathbf{e}_+$  and  $\mathbf{e}_{\mathcal{X}^\perp} = \mathcal{R}\mathbf{e}_1 + \mathcal{R}\mathbf{e}_2$  are  $v_{\pm}$ -independent.

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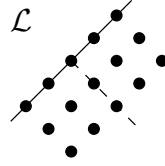
<sup>38</sup>The numbering is slightly inconsistent with §4.1. This choice is more in line with standard conventions: Pauli matrices etc.

### 9.3 The vacuum $\Omega_{\mathcal{X}}$

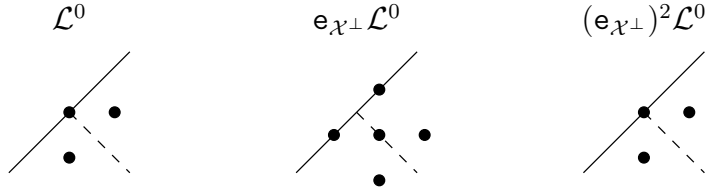
We have

$$\begin{aligned}\Omega_{\mathcal{X}} &= \langle \mathcal{L}^0 \rangle_{\mathcal{X}^\perp} \\ &= \mathcal{L}^0 \oplus \mathbf{e}_{\mathcal{X}^\perp} \mathcal{L}^0 \oplus (\mathbf{e}_{\mathcal{X}^\perp})^2 \mathcal{L}^0\end{aligned}$$

Recall from §7.7 the grading of  $\mathcal{L}$ :

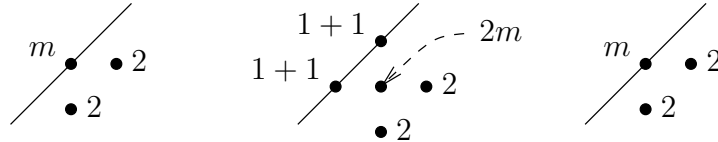


The three components of  $\Omega_{\mathcal{X}}$  are then graded as follows:



Note that  $\mathbf{e}_{\mathcal{X}^\perp}$  shifts parallelly to the dashed line. Exchanging  $V_-$  and  $V_+$  corresponds to reflection about the dashed line.

The  $\mathcal{R}$ -ranks are given by:

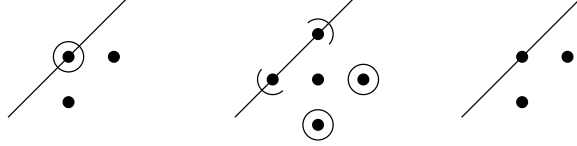


where  $m = 4 + d_{\mathcal{R}}$ . Somewhat informal remarks:

- The three bullets of rank  $m, 2m, m$  have invariant submodules of ranks 4, 8, 4 respectively, obtained by intersecting with  $\mathcal{D}_{\text{vertical}}$ . These vertical submodules have further invariant decompositions.
- The six bullets of rank 2 are contained in  $\mathcal{D}_{\text{vertical}}$  and are irreducible.
- The bullets of rank 1+1 are contained in  $\mathcal{D}_{\text{vertical}}$  and have invariant decomposition  $\mathcal{R} \operatorname{Re} \zeta_{\pm} \oplus \mathcal{R} \operatorname{Im} \zeta_{\pm}$  with  $\zeta_{\pm} \in \mathcal{L}_{\mathbb{C}}^1$  given by  $\zeta_{\pm}(\mathcal{C}) = \zeta_{\pm}(v_{\pm}) = \zeta_{\pm}(\bar{V}) = 0$  and  $\zeta_{+}(v_{-}) = (v_{-} \bar{v}_{+})v_{+}$  and  $\zeta_{-}(v_{+}) = (v_{+} \bar{v}_{-})v_{-}$ .

#### 9.4 Choice of $\mathbf{F}_0\mathcal{L}$

Set  $\mathbf{F}_0\mathcal{L} = \langle \mathbf{F}_0\Omega_{\mathcal{X}} \rangle_{\mathcal{X}}$  where  $\mathbf{F}_0\Omega_{\mathcal{X}} \subseteq \Omega_{\mathcal{X}}$  is the direct sum of:



A circle around a bullet means that we include that entire bullet in  $\mathbf{F}_0\Omega_{\mathcal{X}}$ . The semicircles mean that we include  $\mathcal{R}\operatorname{Re}\zeta_{\pm}$  but not  $\mathcal{R}\operatorname{Im}\zeta_{\pm}$ .

This choice is consistent with §9.1. Note that  $\mathbf{F}_0\mathcal{L} \subseteq \mathbf{G}_{\text{even}}\mathcal{L}$ .

We now check that  $\mathbf{F}_0\mathcal{L}$  is a subalgebra modulo the ideal,

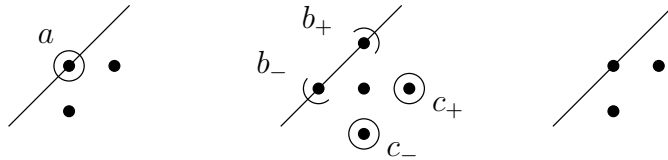
$$[[\mathbf{F}_0\mathcal{L}, \mathbf{F}_0\mathcal{L}]] \subseteq \mathbf{F}_0\mathcal{L} + \mathcal{I}$$

By the bracket estimate in §4.3 it suffices to check two things:

- $\wedge_{\text{tot}}(x(\mathcal{X})) \subseteq \wedge \mathcal{X}$  for all  $x \in \mathbf{F}_0\Omega_{\mathcal{X}}$ .
- $[[\mathbf{F}_0\Omega_{\mathcal{X}}, \mathbf{F}_0\Omega_{\mathcal{X}}]] \subseteq \langle \mathbf{F}_0\Omega_{\mathcal{X}} \rangle_{\mathcal{X}} + \mathcal{I}$ .

The first is by direct calculation<sup>39</sup>. We now discuss the second.

We decompose  $\mathbf{F}_0\Omega_{\mathcal{X}} = a \oplus b \oplus c$  with local abbreviations



and  $b = b_- \oplus b_+$  and  $c = c_- \oplus c_+$ . In particular  $b_{\pm} = \mathcal{R}\operatorname{Re}\zeta_{\pm}$ .

Then  $[[a, a]] \subseteq a$ ,  $[[a, b]] \subseteq b$ ,  $[[a, c]] \subseteq c$ . Also  $[[b, c]] \subseteq \bigoplus_{i,j=2} \mathbf{G}_{i,j}\mathcal{L} \subseteq \langle c \rangle_{\mathcal{X}}$ , by gradings alone. Finally, by direct calculation<sup>40</sup>:

$$\begin{aligned} [[b_{\pm}, b_{\pm}]] &\subseteq \langle b_{\pm} \rangle_{\mathcal{X}} & [[b_-, b_+]] &\subseteq \langle a \rangle_{\mathcal{X}} + \langle b \rangle_{\mathcal{X}} + \mathcal{I}^2 \\ [[c_{\pm}, c_{\pm}]] &\subseteq \langle b_{\pm} \rangle_{\mathcal{X}} & [[c_-, c_+]] &\subseteq \langle a \rangle_{\mathcal{X}} + \mathcal{I}^2 \end{aligned}$$

and there is in particular, and crucially, no  $\mathcal{R}\operatorname{Im}\zeta_{\pm}$  on the right hand sides.

<sup>39</sup>By contrast, this does not hold for all  $x \in \mathcal{R}\operatorname{Im}\zeta_{\pm}$ .

<sup>40</sup>The last of these four is the sample calculation in §B. Minor notational mismatch: here  $c_{\pm}$  are  $\mathcal{R}$ -modules of rank 2; there  $c_{\pm}$  are elements with parameters  $\lambda_{\pm} \in \mathcal{C} = \mathcal{R} \oplus i\mathcal{R}$ .

### 9.5 Choice of $\mathbf{F}_p\mathcal{L}$ for $p > 0$

Set

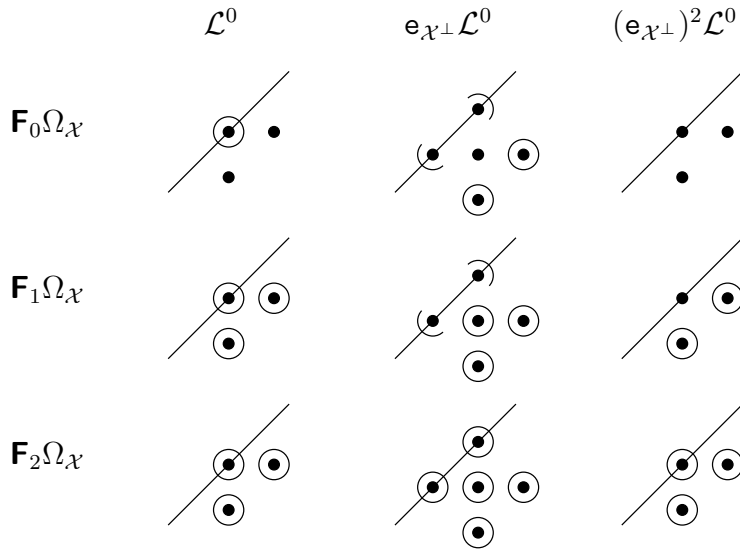
$$\begin{aligned}\mathbf{F}_1\mathcal{L} &= \mathbf{F}_0\mathcal{L} \oplus \mathbf{G}_{\text{odd}}\mathcal{L} \\ \mathbf{F}_2\mathcal{L} &= \mathcal{L} \\ \mathbf{F}_3\mathcal{L} &= \mathcal{L} \\ &\vdots\end{aligned}$$

This choice is consistent with §9.1. We have

$$[\mathbf{F}_p\mathcal{L}, \mathbf{F}_q\mathcal{L}] \subseteq \mathbf{F}_{p+q}\mathcal{L} + \mathcal{I}$$

using §9.4, including  $\mathbf{F}_0\mathcal{L} \subseteq \mathbf{G}_{\text{even}}\mathcal{L}$ .

The definition is equivalent to  $\mathbf{F}_p\mathcal{L} = \langle \mathbf{F}_p\Omega_{\mathcal{X}} \rangle_{\mathcal{X}}$  where:



### 9.6 The 234-condition

By inspection,

$$\begin{aligned}\mathbf{F}_0\mathcal{L} \cap \mathcal{I} &= \langle \bigoplus_{2 \leq |i| \leq 2} \text{Re } \Xi_i \rangle_{\mathcal{X}} \\ \mathbf{F}_1\mathcal{L} \cap \mathcal{I} &= \langle \bigoplus_{1 \leq |i| \leq 2} \text{Re } \Xi_i \rangle_{\mathcal{X}} \\ \mathbf{F}_2\mathcal{L} \cap \mathcal{I} &= \langle \bigoplus_{0 \leq |i| \leq 2} \text{Re } \Xi_i \rangle_{\mathcal{X}} = \mathcal{I}\end{aligned}$$

Hence  $(\mathcal{X}, \mathbf{F}_p\mathcal{L})$  satisfies the 234-condition for all  $p$ .

## 9.7 Filtration of $\mathcal{E}$

Set  $\mathbf{F}_p\mathcal{E} = (\mathbf{F}_p\mathcal{L} + \mathcal{I})/\mathcal{I}$ . It has all the properties in §6.1.

## 10 Remarks that may be helpful when choosing a gauge $G$

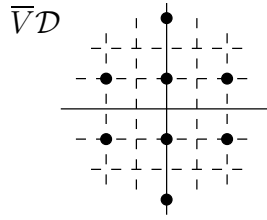
Given is a decomposition  $V = V_- \oplus V_+$ . See (1).

This section is not used in the rest of this paper.

This section uses notation from [RT]. Briefly, by definition, a gauge is a positive definite  $\mathcal{C}$ -Hermitian map  $G : (\overline{V}\mathcal{D}/\mathcal{J}) \times (\overline{V}\mathcal{D}/\mathcal{J}) \rightarrow \mathcal{C}$ . Here  $\mathcal{J} \subseteq \overline{V}\mathcal{D}$  is the subset of  $x$  for which  $x(\mathcal{C}) = x(V) = x(\overline{V} \wedge \overline{V}) = 0$  and for which  $x(\overline{V}) \subseteq \overline{V}\overline{V}$  is symmetric in the two  $\overline{V}$ 's.

### 10.1 Grading on $\overline{V}\mathcal{D}$

Nontrivial components:



The six outer bullets have rank 1. The bullets in the center have rank  $5 + d_{\mathcal{R}}$ .

### 10.2 Decomposition of $\mathcal{J} \subseteq \overline{V}\mathcal{D}$

The decomposition of  $\mathcal{J}$  is obtained similarly to the decomposition of  $\mathcal{I}_{\mathbb{C}}$  in §8.2.

Define  $\theta_z \in \overline{V}\mathcal{D}$  by  $\theta_z(\mathcal{C}) = \theta_z(V) = 0$  and

$$\theta_z(\overline{v_{z'}}) = \overline{(z' - z)} \overline{v_z v_z}$$

Then<sup>41</sup>  $\theta_z \in \mathcal{J}$ . Denote  $\theta_z = \theta_{-\frac{3}{2}} + \theta_{-\frac{1}{2}}\overline{z} + \theta_{\frac{1}{2}}\overline{z}^2 + \theta_{\frac{3}{2}}\overline{z}^3$ . The rank 1 modules  $\Theta_i = \mathcal{C}\theta_i$  are  $v_{\pm}$ -independent. We have

$$\mathcal{J} = \Theta_{-\frac{3}{2}} \oplus \Theta_{-\frac{1}{2}} \oplus \Theta_{\frac{1}{2}} \oplus \Theta_{\frac{3}{2}}$$

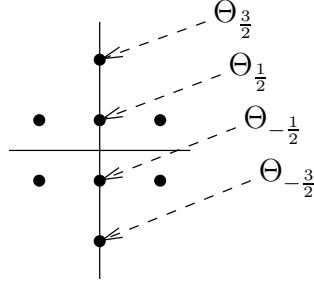
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<sup>41</sup>Note that  $\theta_z(\overline{v_{z'}}\overline{v_{z''}}) = \overline{(z' + z'' - 2z)}\overline{v_z v_z v_z} + \overline{(z' - z)(z'' - z)}\overline{v_z(v_+ v_+ + v_+ v_z)}$ . Since this is symmetric in  $z', z''$  we get  $\theta_z(\overline{V} \wedge \overline{V}) = 0$ , one of the requirements for  $\theta_z \in \mathcal{J}$ .

### 10.3 Complement of $\mathcal{J} \subseteq \bar{V}\mathcal{D}$

It turns out that invariantly associated to (1) is a complement of  $\mathcal{J} \subseteq \bar{V}\mathcal{D}$ .

In the following figure, an arrow means ‘contained in’:



The bullets on the vertical axis are

$$\begin{aligned} \mathbf{G}_{0,+\frac{3}{2}}(\bar{V}\mathcal{D}) &= \Theta_{+\frac{3}{2}} \\ \mathbf{G}_{0,+\frac{1}{2}}(\bar{V}\mathcal{D}) &= \Theta_{+\frac{1}{2}} \oplus \bar{V}_+ \mathbf{G}_{0,0}\mathcal{D} \\ \mathbf{G}_{0,-\frac{1}{2}}(\bar{V}\mathcal{D}) &= \Theta_{-\frac{1}{2}} \oplus \bar{V}_- \mathbf{G}_{0,0}\mathcal{D} \\ \mathbf{G}_{0,-\frac{3}{2}}(\bar{V}\mathcal{D}) &= \Theta_{-\frac{3}{2}} \end{aligned}$$

The remaining four bullets are

$$\mathbf{G}_{-1,\pm\frac{1}{2}}(\bar{V}\mathcal{D}) = \bar{V}_\pm \mathbf{G}_{-1,0}\mathcal{D} \quad \mathbf{G}_{1,\pm\frac{1}{2}}(\bar{V}\mathcal{D}) = \bar{V}_\pm \mathbf{G}_{1,0}\mathcal{D}$$

Combining, we get the internal direct sum decomposition<sup>42</sup>

$$\bar{V}\mathcal{D} = \mathcal{J} \oplus \bar{V}\mathbf{G}_{\text{any},0}\mathcal{D}$$

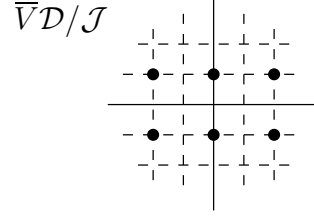
In particular, a gauge  $G$  is determined by its restriction to  $\bar{V}\mathbf{G}_{\text{any},0}\mathcal{D}$ , and every positive definite  $\mathcal{C}$ -Hermitian form on  $\bar{V}\mathbf{G}_{\text{any},0}\mathcal{D}$  determines a  $G$ .

### 10.4 Grading on $\bar{V}\mathcal{D}/\mathcal{J} \simeq \bar{V}\mathbf{G}_{\text{any},0}\mathcal{D}$

The nontrivial components are:

---

<sup>42</sup>To be sure,  $\mathbf{G}_{\text{any},0}\mathcal{D} = \mathbf{G}_{-1,0}\mathcal{D} \oplus \mathbf{G}_{0,0}\mathcal{D} \oplus \mathbf{G}_{1,0}\mathcal{D}$ .



The four corner bullets have rank 1. The two center bullets have rank  $4 + d_{\mathcal{R}}$ .

### 10.5 Remarks

The above discussion suggests conditions that one might impose on  $G$ , and that are invariantly associated to a decomposition (1). Here are two examples:

- One might require that the six bullets in §10.4 be mutually  $G$ -orthogonal.
- One might require that  $G(\overline{v}_- \mathbf{G}_{0,0} \mathcal{L}^0, \overline{v}_- \mathbf{G}_{0,0} \mathcal{L}^0) \subseteq \mathcal{R}$ .  
One might require that  $G(\overline{v}_+ \mathbf{G}_{0,0} \mathcal{L}^0, \overline{v}_+ \mathbf{G}_{0,0} \mathcal{L}^0) \subseteq \mathcal{R}$ .

The second bullet is independent of the choice of  $v_{\pm}$ .

### Structure associated to two compatible decompositions of $V$

Suppose we have two decompositions like (1),

$$\begin{aligned} V &= V'_- \oplus V'_+ \\ &= V''_- \oplus V''_+ \end{aligned} \tag{2a}$$

that are compatible in the sense that

$$\mathbf{r}' \mathbf{r}'' = -\mathbf{r}'' \mathbf{r}' \tag{2b}$$

where  $\mathbf{r}', \mathbf{r}'' \in \text{Aut}_{\mathcal{C}}(V)$  are the involutions defined by

$$\begin{aligned} \mathbf{r}'|_{V'_-} &= -\mathbb{1} & \mathbf{r}''|_{V''_-} &= -\mathbb{1} \\ \mathbf{r}'|_{V'_+} &= \mathbb{1} & \mathbf{r}''|_{V''_+} &= \mathbb{1} \end{aligned}$$



## 11 Reflections

Given are two compatible decompositions  $V = V'_- \oplus V'_+ = V''_- \oplus V''_+$ . See (2).

### 11.1 Equivalent conditions

Condition (2b) is separately equivalent to each of the following:

- $\mathbf{r}'(V''_{\pm}) = V'_{\mp}$ .
- $\mathbf{r}''(V'_{\pm}) = V''_{\mp}$ .
- There exist bases  $V'_{\pm} = \mathcal{C}v'_{\pm}$  and  $V''_{\pm} = \mathcal{C}v''_{\pm}$  such that

$$v'_{\pm} = 2^{-1/2}(v''_{+} \pm v''_{-}) \quad v''_{\pm} = 2^{-1/2}(v'_{+} \pm v'_{-})$$

### 11.2 Compatible bases

We say that  $V'_{\pm} = \mathcal{C}v'_{\pm}$  and  $V''_{\pm} = \mathcal{C}v''_{\pm}$  are compatible if

$$\begin{aligned} v'_{\pm} &= \mathbf{r}''(v''_{\mp}) & v''_{\pm} &= \mathbf{r}'(v'_{\mp}) \\ v'_{\pm} &= 2^{-1/2}(v''_{+} \pm v''_{-}) & v''_{\pm} &= 2^{-1/2}(v'_{+} \pm v'_{-}) \end{aligned}$$

Choosing one basis element determines the other three.

The associated bases for  $(V\overline{V})_{\text{real}}$  defined in §9.2 satisfy

$$c'_0 = c''_0 \quad c'_1 = c''_3 \quad c'_2 = -c''_2 \quad c'_3 = c''_1$$

### 11.3 Associated commuting reflections

Associated to  $\mathbf{r}', \mathbf{r}'' \in \text{Aut}_{\mathcal{C}}(V)$  are automorphisms on various spaces, see §2.3. We collectively denote by  $\mathbf{R}', \mathbf{R}''$  these automorphisms on spaces on which  $-\mathbb{1} \in \text{Aut}_{\mathcal{C}}(V)$  is represented as the identity. Such spaces include  $V\overline{V}$ ,  $\mathcal{D}$ ,  $\mathcal{L}$ ,  $\mathcal{I}$ ,  $\mathcal{E}$ .

Note that  $\mathbf{R}', \mathbf{R}''$  are involutions, and they commute,

$$\mathbf{R}'\mathbf{R}'' = \mathbf{R}''\mathbf{R}'$$

Note that on spaces on which  $-\mathbb{1} \in \text{Aut}_{\mathcal{C}}(V)$  is represented as the identity, both  $\pm\mathbf{r}'$  are represented as  $\mathbf{R}'$ , and both  $\pm\mathbf{r}''$  are represented as  $\mathbf{R}''$ .

## 11.4 Projections

Define the projections

$$\begin{aligned} P_{\text{even}'} &= \frac{1}{2}(\mathbb{1} + R') & P_{\text{even}''} &= \frac{1}{2}(\mathbb{1} + R'') \\ P_{\text{odd}'} &= \frac{1}{2}(\mathbb{1} - R') & P_{\text{odd}''} &= \frac{1}{2}(\mathbb{1} - R'') \end{aligned}$$

They pairwise commute and we often abbreviate  $P_{\text{even}'\text{odd}''} = P_{\text{even}'}P_{\text{odd}''}$  etc.

## 11.5 Lemma

Let  $\mathbf{G}'$  and  $\mathbf{G}''$  be the gradings associated to the decompositions (2a). Then

$$\begin{aligned} P_{\text{even}'}\mathcal{L}_{\mathbb{C}} &= \mathbf{G}'_{\text{even}}\mathcal{L}_{\mathbb{C}} & P_{\text{even}''}\mathcal{L}_{\mathbb{C}} &= \mathbf{G}''_{\text{even}}\mathcal{L}_{\mathbb{C}} \\ P_{\text{odd}'}\mathcal{L}_{\mathbb{C}} &= \mathbf{G}'_{\text{odd}}\mathcal{L}_{\mathbb{C}} & P_{\text{odd}''}\mathcal{L}_{\mathbb{C}} &= \mathbf{G}''_{\text{odd}}\mathcal{L}_{\mathbb{C}} \end{aligned}$$

This is not quite as obvious as may first appear. One can explicitly check that

$$\begin{aligned} P_{\text{even}'(\text{odd}')}V\bar{V} &= \mathbf{G}'_{\text{even}(\text{odd})}V\bar{V} \\ P_{\text{even}'(\text{odd}')}D &= \mathbf{G}'_{\text{even}(\text{odd})}D \end{aligned}$$

which implies the claim.

## 12 Associated decomposition of $\mathcal{I}$

Given are two compatible decompositions  $V = V'_- \oplus V'_+ = V''_- \oplus V''_+$ . See (2). We use bases  $v'_{\pm}$  and  $v''_{\pm}$  that are compatible precisely as in §11.2.

We freely use notation from the discussion of the ideal in §8. All the objects defined there come in two copies, tagged ' and '' respectively.

The discussion is limited to the  $\mathcal{C}$ -module  $\mathcal{I}_*$ . Recall that  $\text{Re} : \mathcal{I}_* \rightarrow \mathcal{I}$  is an  $\mathcal{R}$ -module isomorphism. It commutes with  $R'$  and  $R''$ .

### 12.1 Lemma

We have<sup>43</sup>

$$R'\xi'_z = \xi'_{-z} \qquad R''\xi'_z = z^4\xi'_{1/z}$$

and the same result holds with all ' and '' exchanged.

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<sup>43</sup>One calculates  $R'(\xi'_z(R'(v'_w))) = \dots = \xi'_{-z}(v'_w)$  and  $R''(\xi'_z(R''(v'_w))) = \dots = z^4\xi'_{1/z}(v'_w)$ , using the definition of  $\xi'_z$  in §8. Here  $v'_z = v'_- + zv'_+$  satisfies  $R'v'_z = -v'_{-z}$  and  $R''v'_z = zv'_{1/z}$ .

## 12.2 Lemma

We have  $R'e'_\pm = e'_\pm R'$  but  $R''e'_\pm = e'_{\mp} R''$ , as operators on  $\mathcal{L}_{\mathbb{Q}}$ . Hence<sup>44</sup>

$$P_{\lambda'\lambda''}e'_0 = e'_0P_{\lambda'\lambda''} \quad P_{\lambda'\lambda''}e'_3 = e'_3P_{\lambda'(\lambda''+1)}$$

where  $\lambda' \in \{\text{even}', \text{odd}'\}$  and  $\lambda'' \in \{\text{even}'', \text{odd}''\}$ . Hence for  $k = 3, 4$  we have<sup>45</sup>

$$P_{\lambda'\lambda''}\mathcal{I}_*^k = e'_0P_{\lambda'\lambda''}\mathcal{I}_*^{k-1} + e'_3P_{\lambda'(\lambda''+1)}\mathcal{I}_*^{k-1}$$

which we use below.

## 12.3 Decomposition of $\mathcal{I}_*$

By §12.1 and §12.2 we have

$$\begin{aligned} P_{\text{even}'\text{even}''}\mathcal{I}_*^2 &= \mathcal{C}(\xi'_{-2} + \xi'_2) \oplus \mathcal{C}\xi'_0 \\ P_{\text{even}'\text{odd}''}\mathcal{I}_*^2 &= \mathcal{C}(\xi'_{-2} - \xi'_2) \\ P_{\text{odd}'\text{even}''}\mathcal{I}_*^2 &= \mathcal{C}(\xi'_{-1} + \xi'_1) \\ P_{\text{odd}'\text{odd}''}\mathcal{I}_*^2 &= \mathcal{C}(\xi'_{-1} - \xi'_1) \\ P_{\text{even}'\text{even}''}\mathcal{I}_*^3 &= \mathcal{C}(e'_+\xi'_{-2} + e'_-\xi'_2) \oplus \mathcal{C}e'_0\xi'_0 \\ P_{\text{even}'\text{odd}''}\mathcal{I}_*^3 &= \mathcal{C}(e'_+\xi'_{-2} - e'_-\xi'_2) \oplus \mathcal{C}e'_3\xi'_0 \\ P_{\text{odd}'\text{even}''}\mathcal{I}_*^3 &= \mathcal{C}e'_0(\xi'_{-1} + \xi'_1) \oplus \mathcal{C}e'_3(\xi'_{-1} - \xi'_1) \\ P_{\text{odd}'\text{odd}''}\mathcal{I}_*^3 &= \mathcal{C}e'_3(\xi'_{-1} + \xi'_1) \oplus \mathcal{C}e'_0(\xi'_{-1} - \xi'_1) \\ P_{\text{even}'\text{even}''}\mathcal{I}_*^4 &= 0 \\ P_{\text{even}'\text{odd}''}\mathcal{I}_*^4 &= \mathcal{C}e'_0e'_3\xi'_0 \\ P_{\text{odd}'\text{even}''}\mathcal{I}_*^4 &= \mathcal{C}e'_0e'_3(\xi'_{-1} - \xi'_1) \\ P_{\text{odd}'\text{odd}''}\mathcal{I}_*^4 &= \mathcal{C}e'_0e'_3(\xi'_{-1} + \xi'_1) \end{aligned}$$

Same with all  $P_{\text{even}'\text{odd}''}$  and  $P_{\text{odd}'\text{even}''}$  exchanged on the left hand sides, and with all  $'$  replaced by  $''$  on the right hand sides. In particular,

$$\begin{aligned} P_{\text{even}'\text{even}''}\mathcal{I}_*^2 &= \mathcal{C}(\xi'_{-2} + \xi'_2) \oplus \mathcal{C}\xi'_0 \\ &= \mathcal{C}(\xi''_{-2} + \xi''_2) \oplus \mathcal{C}\xi''_0 \end{aligned}$$

<sup>44</sup>Here  $e'_0 = e'_+ + e'_-$  and  $e'_3 = e'_+ - e'_-$ , instances of  $e'_i = e'_{c'_i}$  with  $c'_i$  defined in §9.2.

<sup>45</sup>Using  $\mathcal{I}_*^k = e'_-\mathcal{I}_*^{k-1} + e'_+\mathcal{I}_*^{k-1} = e'_0\mathcal{I}_*^{k-1} + e'_3\mathcal{I}_*^{k-1}$  for  $k = 3, 4$ .

## 12.4 Lemma

We have<sup>46</sup>

$$\begin{aligned} P_{\text{even}'\text{even}''}\mathcal{I}_*^2 &= \mathcal{C}(\xi'_{-2} + \xi'_2) \oplus \mathcal{C}(\xi''_{-2} + \xi''_2) \\ P_{\text{even}'\text{even}''}\mathcal{I}_*^3 &= e_0 P_{\text{even}'\text{even}''}\mathcal{I}_*^2 \\ P_{\text{even}'\text{even}''}\mathcal{I}_*^4 &= 0 \end{aligned}$$

with  $e_0 = e'_0 = e''_0$ .

## 13 Associated filtration by cubes $_{\mathcal{E}}$

Given are two compatible decompositions  $V = V'_- \oplus V'_+ = V''_- \oplus V''_+$ . See (2). We use bases  $v'_\pm$  and  $v''_\pm$  that are compatible precisely as in §11.2.

We use notation from our discussion of the filtration in §9. All objects defined there come in two copies, tagged ' and '' respectively. For example,  $\mathbf{F}'$  and  $\mathbf{F}''$ .

We construct a filtration  $\mathbf{f}$  as in §6.1, indexed by  $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ . We use §6.3.

### 13.1 Choice of $\mathcal{X}$

Define  $\mathcal{X}'$  and  $\mathcal{X}''$  as in §9.2. Then define

$$\begin{aligned} \mathcal{X} &= \mathcal{X}' \cap \mathcal{X}'' \\ &= \mathcal{R}_{c_0} \end{aligned}$$

with  $c_0 = c'_0 = c''_0 \in (V\overline{V})_{\text{positive}}$ . It satisfies all conditions in §4.1.

### 13.2 Choice of $\mathbf{f}_{p'p''}\mathcal{E}$

For all  $p', p'' \in \mathbb{Z}_{\geq 0}$  set

$$\mathbf{f}_{p'p''}\mathcal{E} = \mathbf{F}'_{p'}\mathcal{E} \cap \mathbf{F}''_{p''}\mathcal{E}$$

Recall that  $\mathbf{F}'_{p'}\mathcal{E}$  is an  $\mathcal{X}'$ -cubes $_{\mathcal{E}}$ , and  $\mathbf{F}''_{p''}\mathcal{E}$  is an  $\mathcal{X}''$ -cubes $_{\mathcal{E}}$ . Therefore they are  $\mathcal{X}$ -cubes $_{\mathcal{E}}$ . But we have yet to see that  $\mathbf{f}_{p'p''}\mathcal{E}$  is an  $\mathcal{X}$ -cube $_{\mathcal{E}}$ .

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<sup>46</sup>First equation: It essentially suffices to check that the two summands have trivial intersection; take  $2e'_-e'_+ = e'_0e'_3 = e''_0e''_1$  which annihilates the first summand but not the second.

### 13.3 Lemma

The following spaces are closed under  $R'$  and  $R''$ :

- The ideal  $\mathcal{I}$ .
- The representatives  $\mathbf{F}'_{p'}\mathcal{L}$  and  $\mathbf{F}''_{p''}\mathcal{L}$  defined in §9.
- Intersections and sums of these spaces.

The first follows from §2.3. The second follows from §9.1 and §11.1.

### 13.4 Proof that $\mathbf{f}_{p'p''}\mathcal{E}$ is an $\mathcal{X}$ -cube $_{\mathcal{E}}$

Set  $\mathcal{L}' = \mathbf{F}'_{p'}\mathcal{L}$  and  $\mathcal{L}'' = \mathbf{F}''_{p''}\mathcal{L}$ , keeping in mind the implicit dependence on  $p', p''$ . These  $\mathcal{X}$ -cubes $_{\mathcal{L}}$  are representatives of  $\mathbf{F}'_{p'}\mathcal{E}$  and  $\mathbf{F}''_{p''}\mathcal{E}$ . We prove that their intersection  $\mathcal{L}' \cap \mathcal{L}''$  is a representative of  $\mathbf{f}_{p'p''}\mathcal{E}$ .

By the 2nd bullet in §4.11 it suffices to check the 2nd bullet in §3.1, that is

$$(\mathcal{L}' + \mathcal{L}'') \cap \mathcal{I} \subseteq (\mathcal{L}' \cap \mathcal{I}) + (\mathcal{L}'' \cap \mathcal{I})$$

All terms and subterms are closed under  $P_{\lambda'\lambda''}$  by §13.3, and therefore it suffices to show for all  $p', p'', \lambda', \lambda''$  the statement  $\mathbf{l}_{p'p''\lambda'\lambda''}$  defined by

$$\mathbf{l}_{p'p''\lambda'\lambda''} \iff (P\mathcal{L}' + P\mathcal{L}'') \cap P\mathcal{I} \subseteq (P\mathcal{L}' \cap P\mathcal{I}) + (P\mathcal{L}'' \cap P\mathcal{I})$$

Here and below  $P = P_{\lambda'\lambda''}$ , keeping in mind the implicit dependence on  $\lambda', \lambda''$ . We show below that one is always in one of the following three sufficient situations: either  $P\mathcal{L}' = 0$ ; or  $P\mathcal{L}'' = 0$ ; or  $P\mathcal{I} \subseteq (P\mathcal{L}' \cap P\mathcal{I}) + (P\mathcal{L}'' \cap P\mathcal{I})$ .

**Proof of  $\mathbf{l}_{p'p''\text{even}'\text{even}''}$ .**  $P\mathcal{I} \subseteq P\langle \text{Re } \Xi'_{-2} \oplus \text{Re } \Xi'_2 \rangle_{\mathcal{X}'} + P\langle \text{Re } \Xi''_{-2} \oplus \text{Re } \Xi''_2 \rangle_{\mathcal{X}''}$  by §12.4. Since  $P\langle \text{Re } \Xi'_{-2} \oplus \text{Re } \Xi'_2 \rangle_{\mathcal{X}'} \subseteq P\mathcal{L}' \cap P\mathcal{I}$  by §9.6, and similar for  $\mathcal{X}''$ , we conclude that  $P\mathcal{I} \subseteq (P\mathcal{L}' \cap P\mathcal{I}) + (P\mathcal{L}'' \cap P\mathcal{I})$ .

**Proof of  $\mathbf{l}_{p'p''\text{odd}'\lambda''}$ .** Distinguish two cases:

- If  $p' = 0$  then  $P\mathcal{L}' = 0$ .  
Use  $\mathcal{L}' \subseteq \mathbf{G}'_{\text{even}}\mathcal{L}$  from §9.4, and §11.5.
- If  $p' \geq 1$  then  $P\mathcal{I} \subseteq P\mathcal{L}' \cap P\mathcal{I}$ .  
Use  $\mathbf{G}'_{\text{odd}}\mathcal{L} \subseteq \mathcal{L}'$  from §9.5, and  $P\mathcal{I} \subseteq P\mathcal{L} \subseteq P_{\text{odd}'}\mathcal{L} \subseteq \mathbf{G}'_{\text{odd}}\mathcal{L}$  using §11.5.

**Proof of  $\mathbf{l}_{p'p''\lambda'\text{odd}''}$ .** Analogous.

### 13.5 Lemma

Recall the notation  $\langle \sigma \rangle_{\mathcal{X}} = \sigma + \mathbf{e}_0 \sigma$  where  $\mathbf{e}_0 = \mathbf{e}'_0 = \mathbf{e}''_0$ .

We have, with the abbreviation  $\sigma_{\text{even}'\text{even}''} = \text{Re}(\mathcal{C}(\xi'_{-2} + \xi'_2)) \subseteq \mathbf{P}_{\text{even}'\text{even}''}\mathcal{I}^2$ ,

$$\begin{aligned} \mathbf{F}'_0 \mathcal{L} \cap \mathbf{F}''_0 \mathcal{L} \cap \mathcal{I} &= 0 \\ \mathbf{F}'_0 \mathcal{L} \cap \mathbf{F}''_1 \mathcal{L} \cap \mathcal{I} &= \langle \mathbf{P}_{\text{even}'\text{odd}''}\mathcal{I}^2 \rangle_{\mathcal{X}} \\ \mathbf{F}'_0 \mathcal{L} \cap \mathbf{F}''_2 \mathcal{L} \cap \mathcal{I} &= \langle \sigma_{\text{even}'\text{even}''} \rangle_{\mathcal{X}} \oplus \langle \mathbf{P}_{\text{even}'\text{odd}''}\mathcal{I}^2 \rangle_{\mathcal{X}} \\ \mathbf{F}'_1 \mathcal{L} \cap \mathbf{F}''_1 \mathcal{L} \cap \mathcal{I} &= \langle \mathbf{P}_{\text{even}'\text{odd}''}\mathcal{I}^2 \rangle_{\mathcal{X}} \oplus \langle \mathbf{P}_{\text{odd}'\text{even}''}\mathcal{I}^2 \rangle_{\mathcal{X}} \oplus \mathbf{P}_{\text{odd}'\text{odd}''}\mathcal{I} \\ \mathbf{F}'_1 \mathcal{L} \cap \mathbf{F}''_2 \mathcal{L} \cap \mathcal{I} &= \langle \sigma_{\text{even}'\text{even}''} \rangle_{\mathcal{X}} \oplus \langle \mathbf{P}_{\text{even}'\text{odd}''}\mathcal{I}^2 \rangle_{\mathcal{X}} \oplus \mathbf{P}_{\text{odd}'\text{even}''}\mathcal{I} \oplus \mathbf{P}_{\text{odd}'\text{odd}''}\mathcal{I} \\ \mathbf{F}'_2 \mathcal{L} \cap \mathbf{F}''_2 \mathcal{L} \cap \mathcal{I} &= \langle \mathbf{P}_{\text{even}'\text{even}''}\mathcal{I}^2 \rangle_{\mathcal{X}} \oplus \mathbf{P}_{\text{even}'\text{odd}''}\mathcal{I} \oplus \mathbf{P}_{\text{odd}'\text{even}''}\mathcal{I} \oplus \mathbf{P}_{\text{odd}'\text{odd}''}\mathcal{I} \end{aligned}$$

This follows from the formulas in §9.6 and from §12.3, §12.4.

### 13.6 Lemma

If  $(\lambda', \lambda'') \neq (\text{even}', \text{even}'')$  then there is a  $Y \subseteq \mathbf{P}\mathcal{I}^3$  such that

$$\mathbf{P}\mathcal{I} = \langle \mathbf{P}\mathcal{I}^2 \rangle_{\mathcal{X}} \oplus Y \oplus \mathbf{P}\mathcal{I}^4 \quad (3a)$$

$$\mathbf{P}\mathcal{L} = \langle \mathbf{P}\Omega_{\mathcal{X}, \leq 2} \rangle_{\mathcal{X}} \oplus Y \oplus \mathbf{P}\mathcal{I}^4 \quad (3b)$$

where  $\mathbf{P} = \mathbf{P}_{\lambda'\lambda''}$  and where  $\Omega_{\mathcal{X}, \leq 2} = \Omega_{\mathcal{X}} \cap (\mathcal{L}^0 \oplus \mathcal{L}^1 \oplus \mathcal{L}^2)$ .

To check this, first note that  $\mathbf{P}\mathcal{L} = \langle \mathbf{P}\Omega_{\mathcal{X}} \rangle_{\mathcal{X}} = \langle \mathbf{P}\Omega_{\mathcal{X}, \leq 2} \rangle_{\mathcal{X}} \oplus \langle Z \rangle_{\mathcal{X}}$  where  $Z = \mathbf{P}(\Omega_{\mathcal{X}} \cap \mathcal{L}^3)$ . It is easy to see that  $\text{rank}_{\mathcal{R}} Z = 2$  given  $(\lambda', \lambda'') \neq (\text{even}', \text{even}'')$ . Hence  $Z = \mathbf{i}_0(\mathbf{P}\mathcal{I}^4)$ , essentially because  $\supseteq$  and the right hand side has rank 2. Hence<sup>47</sup>  $\langle Z \rangle_{\mathcal{X}} = \mathbf{i}_0(\mathbf{P}\mathcal{I}^4) \oplus \mathbf{P}\mathcal{I}^4$ . But  $\mathbf{i}_0(\mathbf{P}\mathcal{I}^4) \not\subseteq \mathcal{I}^3$  cannot be taken to be  $Y$ . Instead take  $Y = \text{Re}(\mathcal{C}(\mathbf{e}'_3 \xi'_0))$  or  $Y = \text{Re}(\mathcal{C}(\mathbf{e}'_3(\xi'_{-1} \pm \xi'_1)))$ , depending on  $\lambda'\lambda''$ ; then (3a) follows from §12.3. For (3b) it essentially suffices, since  $Y \subseteq \mathbf{P}\mathcal{L}^3$  has rank 2, to check that it does not intersect  $\langle \mathbf{P}\Omega_{\mathcal{X}, \leq 2} \rangle_{\mathcal{X}} \cap \mathcal{L}^3$ . But applying  $\mathbf{e}'_0$  to the latter gives zero, whereas  $\mathbf{e}'_0 Y = \mathbf{P}\mathcal{I}^4$  has rank 2.

Corollary: If  $(\lambda', \lambda'') \neq (\text{even}', \text{even}'')$  then the  $\mathcal{X}$ -cube $_{\mathcal{L}}$   $S = \langle \mathbf{P}\Omega_{\mathcal{X}, \leq 2} \rangle_{\mathcal{X}}$  satisfies  $\mathbf{P}\mathcal{L} = S + \mathbf{P}\mathcal{I}$ , and  $(\mathcal{X}, S)$  satisfies the 234-condition.

To see that the 234-condition holds, note that  $\mathbf{P}\mathcal{I}^2 \subseteq \mathbf{P}\mathcal{L}^2 \subseteq S$ . Hence  $\langle \mathbf{P}\mathcal{I}^2 \rangle_{\mathcal{X}} \subseteq S$ , and then (3) implies  $S \cap \mathbf{P}\mathcal{I} = \langle \mathbf{P}\mathcal{I}^2 \rangle_{\mathcal{X}}$ .

<sup>47</sup>Use the anticommutation relation  $\mathbf{i}_0 \mathbf{e}_0 + \mathbf{e}_0 \mathbf{i}_0 = (\text{invertible})\mathbb{1}$ .

### 13.7 The 234-condition

By §13.4,  $S_0 = \mathbf{F}'_{p'}\mathcal{L} \cap \mathbf{F}''_{p''}\mathcal{L}$  is a representative of the  $\mathcal{X}$ -cube $_{\mathcal{E}}$   $\mathbf{f}_{p'p''}\mathcal{E}$ . We construct another representative  $S$  such that  $(\mathcal{X}, S)$  satisfies the 234-condition:

- If  $p' = 0$  or  $p'' = 0$  then we can take  $S = S_0$  by §13.5.
- If  $p' = p'' = 1$  then  $P_{\text{odd}'\text{odd}''}S_0 = P_{\text{odd}'\text{odd}''}\mathcal{L}$ . Set

$$S = P_{\text{even}'\text{even}''}S_0 \oplus P_{\text{even}'\text{odd}''}S_0 \oplus P_{\text{odd}'\text{even}''}S_0 \oplus \langle P_{\text{odd}'\text{odd}''}\Omega_{\mathcal{X}, \leq 2} \rangle_{\mathcal{X}}$$

using notation from §13.6. This is a representative,  $S_0 + \mathcal{I} = S + \mathcal{I}$ , because  $PS_0 + P\mathcal{I} = PS + P\mathcal{I}$  with  $P = P_{\text{odd}'\text{odd}''}$ , by the corollary in §13.6; both sides are equal to  $P\mathcal{L}$ . Use the same corollary to check the P-part of the 234-condition for  $(\mathcal{X}, S)$ ; for the other three parts use §13.5.

- If  $p' = 1$  and  $p'' = 2$  then  $P_{\text{odd}'}S_0 = P_{\text{odd}'}\mathcal{L}$ . In this case set  $S$  equal to

$$P_{\text{even}'\text{even}''}S_0 \oplus P_{\text{even}'\text{odd}''}S_0 \oplus \langle P_{\text{odd}'\text{even}''}\Omega_{\mathcal{X}, \leq 2} \rangle_{\mathcal{X}} \oplus \langle P_{\text{odd}'\text{odd}''}\Omega_{\mathcal{X}, \leq 2} \rangle_{\mathcal{X}}$$

- If  $p' = p'' = 2$  then  $S_0 = \mathcal{L}$ . In this case set  $S$  equal to

$$P_{\text{even}'\text{even}''}S_0 \oplus \langle P_{\text{even}'\text{odd}''}\Omega_{\mathcal{X}, \leq 2} \rangle_{\mathcal{X}} \oplus \langle P_{\text{odd}'\text{even}''}\Omega_{\mathcal{X}, \leq 2} \rangle_{\mathcal{X}} \oplus \langle P_{\text{odd}'\text{odd}''}\Omega_{\mathcal{X}, \leq 2} \rangle_{\mathcal{X}}$$

### 13.8 Filtration of $\mathcal{E}$

The filtration  $\mathbf{f}$  has all the properties in §6.1.

### Some explicit calculations

The filtration  $\mathbf{f}$  in §13 can be used to set up a filtered expansion, see the introduction §1. The following sections contain the results of explicit calculations.

We use bases  $v'_{\pm}$  and  $v''_{\pm}$  compatible as in §11.2. We abbreviate  $v_{\pm} = v'_{\pm}$ .

## 14 Notation

### 14.1 Involution $\mathbf{x}$

Define  $\mathbf{x} \in \text{Aut}_{\mathcal{C}}(V)$  by, equivalently,

- $\mathbf{x} = 2^{-1/2}(\mathbf{r}' + \mathbf{r}'')$ .
- $v''_{\pm} = \mathbf{x}(v'_{\pm})$ .
- $v'_{\pm} = \mathbf{x}(v''_{\pm})$ .

Then  $\mathbf{x}^2 = \mathbb{1}$  and  $\mathbf{r}'' = \mathbf{x}\mathbf{r}'\mathbf{x}$ .

## 14.2 Third reflection $\mathbf{r}'''$

Define  $\mathbf{r}''' \in \text{Aut}_{\mathcal{C}}(V)$  by

$$\mathbf{r}''' = i\mathbf{r}'\mathbf{r}''$$

It follows immediately that:

- The  $\mathbf{r}', \mathbf{r}'', \mathbf{r}'''$  anticommute pairwise.
- $(\mathbf{r}')^2 = (\mathbf{r}'')^2 = (\mathbf{r}''')^2 = \mathbb{1}$ .  
 $\mathbf{r}''\mathbf{r}''' = -i\mathbf{r}'$  and  $\mathbf{r}'''\mathbf{r}' = -i\mathbf{r}''$  and  $\mathbf{r}'\mathbf{r}'' = -i\mathbf{r}'''$ .
- The sixteen elements  $\lambda\mathbb{1}, \lambda\mathbf{r}', \lambda\mathbf{r}'', \lambda\mathbf{r}'''$  with  $\lambda = \pm 1, \pm i$  are pairwise distinct and form a subgroup of  $\text{Aut}_{\mathcal{C}}(V)$ .

## 14.3 Involutions $\mathbf{x}', \mathbf{x}''$

Define  $\mathbf{x}', \mathbf{x}'' \in \text{Aut}_{\mathcal{C}}(V)$  by

$$\begin{aligned}\mathbf{x}' &= 2^{-1/2}(\mathbf{r}' + \mathbf{r}''') \\ \mathbf{x}'' &= 2^{-1/2}(\mathbf{r}'' - \mathbf{r}''')\end{aligned}$$

Then  $\mathbf{x}'\mathbf{r}' = \mathbf{r}'''\mathbf{x}'$  and  $\mathbf{x}''\mathbf{r}'' = -\mathbf{r}'''\mathbf{x}''$  and these are involutions,  $(\mathbf{x}')^2 = (\mathbf{x}'')^2 = \mathbb{1}$ . We have chosen signs such that  $\mathbf{x}\mathbf{x}' = \mathbf{x}''\mathbf{x}$ .

## 14.4 Associated automorphisms

Associated to  $\mathbf{r}', \mathbf{r}'', \mathbf{r}''', \mathbf{x}, \mathbf{x}', \mathbf{x}'' \in \text{Aut}_{\mathcal{C}}(V)$  are automorphisms on various other spaces, see §2.3. We collectively denote by  $\mathbf{R}', \mathbf{R}'', \mathbf{R}''', \mathbf{X}, \mathbf{X}', \mathbf{X}''$  these automorphisms on spaces on which  $-\mathbb{1} \in \text{Aut}_{\mathcal{C}}(V)$  is represented as the identity.



## 15 On the naive leading term $\gamma_{(0)}$

Part of this section appears again in §16. Calculations are not written out explicitly. Recall that we abbreviate  $v_{\pm} = v'_{\pm}$ .

### 15.1 Lemma

The following map is an  $\mathcal{R}$ -module isomorphism:

$$\begin{aligned} \text{Der}(\mathcal{R}) \oplus \mathcal{R}^6 &\rightarrow \mathbf{f}_{00}\mathcal{E}^1 \\ \xi_a \oplus a &\mapsto \gamma_{(0)} \end{aligned}$$

where

$$\begin{aligned} \gamma_{(0)}|_{\mathcal{R}} &= (v_+\overline{v_+} + v_-\overline{v_-})\xi_a \\ \gamma_{(0)}(v_-) &= \frac{1}{2}(-\frac{1}{2}a_2 - \frac{1}{2}a_3 + ia_4 - a_5)v_+\overline{v_+}v_- \\ &\quad + \frac{1}{2}(-a_1 - \frac{1}{2}a_2 - \frac{1}{2}a_3 + ia_4 - a_5 + ia_6)v_-\overline{v_-}v_- \\ &\quad + \frac{1}{2}(-\frac{1}{2}a_2 - \frac{1}{2}a_3)v_-\overline{v_+}v_+ + \frac{1}{2}(\frac{1}{2}a_2 - \frac{1}{2}a_3 + ia_6)v_+\overline{v_-}v_+ \\ \gamma_{(0)}(v_+) &= (\text{defined such that } R''\gamma_{(0)} = \gamma_{(0)}) \end{aligned}$$

One can check that  $R'\gamma_{(0)} = R''\gamma_{(0)} = \gamma_{(0)}$ . Applying  $X$  on the right hand side of the map corresponds to exchanging  $a_3, a_1$  on the left hand side of the map.

### 15.2 Lemma

For every  $\gamma_{(0)} \in \mathbf{f}_{00}\mathcal{E}^1$  we have  $[\gamma_{(0)}, \gamma_{(0)}] = 0$  if and only if

$$\begin{aligned} 0 &= -\xi_a(a_1) + (a_6)^2 + a_5a_1 \\ 0 &= -\xi_a(a_2) - (a_6)^2 + a_5a_2 \\ 0 &= -\xi_a(a_3) + (a_6)^2 + a_5a_3 \\ 0 &= -\xi_a(a_6) + a_2a_6 + a_5a_6 \end{aligned}$$

and

$$0 = a_2a_3 + a_3a_1 + a_1a_2 - (a_6)^2$$

### 15.3 An explicit solution

Let  $\mathcal{R}$  be the smooth real functions on  $\mathbb{R}^4$ . Let  $\partial_1, \partial_2, \partial_3, \partial_4$  be the partial derivatives. Let  $\tau$  be the first coordinate, so that  $(\partial_1\tau, \partial_2\tau, \dots) = (1, 0, 0, 0)$ . Let  $u \in \mathcal{R}$  be any function that has a multiplicative inverse  $u^{-1} \in \mathcal{R}$ , and that depends only on the second coordinate,  $(\partial_1u, \partial_2u, \dots) = (0, *, 0, 0)$ . Then

$$\begin{aligned} \xi_a = \partial_1 \quad & a_1 = -u + \tanh \tau & a_4 = 0 \\ & a_2 = -\tanh \tau & a_5 = 0 \\ & a_3 = -u^{-1} + \tanh \tau & a_6 = 1/(\cosh \tau) \end{aligned}$$

solves all the equations in §15.2.

Informally, the above solution is general up to exceptional cases, with the understanding that one can obtain other solutions from the one above by changing coordinates or rescaling  $v_{\pm}$ . We make no attempt to make this precise.

### 15.4 Nonzero 2nd cohomology for the explicit solution in §15.3

Define the differential  $d_{(0)} = [\gamma_{(0)}, \cdot] : \mathcal{E} \rightarrow \mathcal{E}$ :

$$0 \xrightarrow{d_{(0)}} \mathcal{E}^0 \xrightarrow{d_{(0)}} \mathcal{E}^1 \xrightarrow{d_{(0)}} \mathcal{E}^2 \xrightarrow{d_{(0)}} \mathcal{E}^3 \xrightarrow{d_{(0)}} \mathcal{E}^4 \xrightarrow{d_{(0)}} 0$$

Then the 2nd cohomology is nonzero, in fact one has the stronger statement: there are two solutions to the linearized equations,  $x, y \in \ker(d_{(0)}|_{\mathcal{E}^1})$ , whose bracket is nonzero in the 2nd cohomology,  $[x, y] \notin \text{image}(d_{(0)}|_{\mathcal{E}^1})$ .

By direct calculation, these  $x, y \in \mathcal{L}^1$  solve the linearized equations<sup>48,49</sup>:

- $x|_{\mathcal{R}} = \exp(-\frac{1}{2}u^{-1}\tau)(v_+\overline{v_+} - v_-\overline{v_-})\partial_3$  and  $x(v_{\pm}) = 0$ .
- $y|_{\mathcal{R}} = 0$  and

$$\begin{aligned} y(v_-) &= \exp(-u\tau)(1 - iu \cosh \tau)(v_+\overline{v_+}v_- + v_-\overline{v_+}v_+) \\ &\quad + \exp(-u\tau)(v_-\overline{v_-}v_- + v_+\overline{v_-}v_+) \\ y(v_+) &= (\text{defined such that } R''y = y) \end{aligned}$$

<sup>48</sup>As an aside,  $x \in \mathfrak{f}_{01}\mathcal{E}^1$  and  $R'x = -R''x = x$ , and  $y \in \mathfrak{f}_{20}\mathcal{E}^1$  and  $R'y = R''y = y$ .

<sup>49</sup>As an aside, note that the set of solutions to the linearized equations,  $\ker(d_{(0)}|_{\mathcal{E}^1})$ , is closed under multiplication by elements of  $\mathcal{R}$  that are annihilated by  $\partial_1$ .

To check that  $[x, y]$  is nonzero in the 2nd cohomology, let  $f \in \mathcal{R}$  denote the third coordinate,  $(\partial_1 f, \partial_2 f, \dots) = (0, 0, 1, 0)$ . We use the following lemma:

- $(e_- e_+ \llbracket \gamma_{(0)}, \mathcal{L}^1 \rrbracket)(f) = 0$ , where  $e_{\pm}$  are defined as in §8.1.

On the other hand,  $(e_- e_+ \llbracket x, y \rrbracket)(f) \neq 0$  by direct verification. Since all elements of  $\mathcal{I}^2$  annihilate  $\mathcal{R}$ , we conclude that  $[x, y] \notin \text{image}(d_{(0)}|_{\mathcal{E}^1})$ .

## 16 On the leading term $\gamma_0$

We dump the result of computer calculations of the equations  $[\gamma_0, \gamma_0] = 0$  that the leading term  $\gamma_0 \in \mathcal{P}^1/s\mathcal{P}^1$  has to satisfy. Here  $\mathcal{P}$  is the Rees algebra of  $\mathbf{f}$ .

We do not suggest that one should explicitly work with the equations as presented here. The discussion is incomplete in many ways, in particular we do not consider  $\mathcal{P}^k/s\mathcal{P}^k$  for  $k = 0, 3, 4$ , which would be necessary to discuss the symmetries and identities enjoyed by these equations.

Even so, there are interesting things to see here. For example in §16.15 we point out certain degeneracies that are very closely related to §15.4.

### 16.1 Ranks of the components of $\mathcal{P}/s\mathcal{P}$

The  $\mathcal{R}$ -ranks of the components of  $\mathcal{P}/s\mathcal{P}$  are as follows:

	multiplicity	$\mathcal{E}^0$	$\mathcal{E}^1$	$\mathcal{E}^2$	$\mathcal{E}^3$	$\mathcal{E}^4$
$\mathbf{f}_{00}$	1	$d_{\mathcal{R}} + 2$	$d_{\mathcal{R}} + 6$	5	1	0
$\mathbf{f}_{10}/\mathbf{f}_{<10}$	2	2	$d_{\mathcal{R}} + 7$	$d_{\mathcal{R}} + 6$	1	0
$\mathbf{f}_{20}/\mathbf{f}_{<20}$	2	0	1	1	0	0
$\mathbf{f}_{11}/\mathbf{f}_{<11}$	1	2	$d_{\mathcal{R}} + 8$	$2d_{\mathcal{R}} + 10$	$d_{\mathcal{R}} + 4$	0
$\mathbf{f}_{21}/\mathbf{f}_{<21}$	2	0	1	$d_{\mathcal{R}} + 4$	$d_{\mathcal{R}} + 3$	0
$\mathbf{f}_{22}/\mathbf{f}_{<22}$	1	0	0	1	$d_{\mathcal{R}} + 3$	$d_{\mathcal{R}} + 2$

Here rank  $md_{\mathcal{R}} + n$  means that the respective component is  $\simeq \text{Der}(\mathcal{R})^m \oplus \mathcal{R}^n$ . For example,  $\mathbf{f}_{10}\mathcal{E}^1/\mathbf{f}_{<10}\mathcal{E}^1 \simeq \text{Der}(\mathcal{R}) \oplus \mathcal{R}^7$ , and the multiplicity reminds us that there is also the isomorphic  $\mathbf{f}_{01}\mathcal{E}^1/\mathbf{f}_{<01}\mathcal{E}^1$ .

## 16.2 Summary of conventions

We abbreviate  $v_{\pm} = v'_{\pm}$ . Hence

$$\begin{aligned} v'_{\pm} &= v_{\pm} \\ v''_{\pm} &= 2^{-1/2}(v_{+} \pm v_{-}) \end{aligned}$$

We use the following definitions, consistent with the rest of this paper:

$$\begin{aligned} \mathbf{r}'(v_{\pm}) &= \pm v_{\pm} \\ \mathbf{r}''(v_{\pm}) &= v_{\mp} \\ \mathbf{x} &= 2^{-1/2}(\mathbf{r}' + \mathbf{r}'') \\ \mathbf{x}' &= 2^{-1/2}(\mathbf{r}' + i\mathbf{r}'\mathbf{r}'') \\ \mathbf{x}'' &= 2^{-1/2}(\mathbf{r}' + i\mathbf{r}''\mathbf{r}') \end{aligned}$$

We denote by  $\mathbf{R}', \mathbf{R}'', \mathbf{X}, \mathbf{X}', \mathbf{X}''$  the associated elements of  $\text{Aut}(\mathcal{E})$ .

## 16.3 The auxiliary map $\gamma^0$

Define

$$\begin{aligned} \text{Der}(\mathcal{R}) \oplus \mathcal{R}^5 &\rightarrow \text{P}_{\text{even}'\text{even}''}\mathcal{E}^1 \\ \xi \oplus x &\mapsto \gamma_{\xi,x}^0 = \gamma^0 \end{aligned}$$

by

$$\begin{aligned} \gamma^0|_{\mathcal{R}} &= (v_{+}\overline{v_{+}} + v_{-}\overline{v_{-}})\xi \\ \gamma^0(v_{-}) &= \frac{1}{2}(-\frac{1}{2}x_2 - \frac{1}{2}x_3 + ix_4 - x_5)v_{+}\overline{v_{+}}v_{-} \\ &\quad + \frac{1}{2}(-x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 + ix_4 - x_5)v_{-}\overline{v_{-}}v_{-} \\ &\quad + \frac{1}{2}(-\frac{1}{2}x_2 - \frac{1}{2}x_3)v_{-}\overline{v_{+}}v_{+} + \frac{1}{2}(\frac{1}{2}x_2 - \frac{1}{2}x_3)v_{+}\overline{v_{-}}v_{+} \\ \gamma^0(v_{+}) &= (\text{defined such that } \mathbf{R}''\gamma^0 = \gamma^0) \end{aligned}$$

One can check that applying  $\mathbf{X}$  on the right hand side of the map corresponds to exchanging  $x_3, x_1$  on the left hand side of the map. Applying  $\mathbf{X}'$  corresponds to exchanging  $x_1, x_2$ . Applying  $\mathbf{X}''$  corresponds to exchanging  $x_2, x_3$ .

## 16.4 The auxiliary map $\gamma^I$

Define

$$\begin{aligned} \text{Der}(\mathcal{R}) \oplus \mathcal{R}^7 &\rightarrow \mathcal{P}_{\text{even'odd''}} \mathcal{E}^1 \\ \xi \oplus x &\mapsto \gamma_{\xi,x}^I = \gamma^I \end{aligned}$$

by

$$\begin{aligned} \gamma^I|_{\mathcal{R}} &= (v_+ \overline{v_+} - v_- \overline{v_-}) \xi \\ \gamma^I(v_-) &= \frac{1}{2}(-ix_1 - x_6 + ix_7)v_+ \overline{v_+} v_- \\ &\quad + (x_4 + ix_5)v_+ \overline{v_-} v_+ \\ &\quad + \frac{1}{2}(x_2 - x_3)v_- \overline{v_+} v_+ \\ &\quad + \frac{1}{2}(ix_1 - x_2 - x_3 - x_6 + ix_7)v_- \overline{v_-} v_- \\ \gamma^I(v_+) &= (\text{defined such that } R''\gamma^I = -\gamma^I) \end{aligned}$$

One can check that applying  $X''$  on the right hand side of the map corresponds to multiplying  $\xi, x_1, x_2, x_3, x_6, x_7$  by  $-1$  on the left hand side.

## 16.5 The auxiliary map $\gamma^{II}$

Define

$$\begin{aligned} \mathcal{R} &\rightarrow \mathcal{P}_{\text{even'even''}} \mathcal{E}^1 \\ x &\mapsto \gamma_x^{II} = \gamma^{II} \end{aligned}$$

by

$$\begin{aligned} \gamma^{II}|_{\mathcal{R}} &= 0 \\ \gamma^{II}(v_-) &= \frac{1}{2}ixv_+ \overline{v_+} v_- + \frac{1}{2}ixv_- \overline{v_+} v_+ \\ \gamma^{II}(v_+) &= (\text{defined such that } R''\gamma^{II} = \gamma^{II}) \end{aligned}$$

One can check that  $X''\gamma^{II} = \gamma^{II}$ .

## 16.6 The auxiliary map $\gamma^{III}$

Define

$$\begin{aligned} \mathcal{R} &\rightarrow \mathcal{P}_{\text{even'odd''}} \mathcal{E}^1 \\ x &\mapsto \gamma_x^{III} = \gamma^{III} \end{aligned}$$

by

$$\begin{aligned}\gamma^{\text{III}}|_{\mathcal{R}} &= 0 \\ \gamma^{\text{III}}(v_-) &= ixv_- \overline{v_+} v_+ \\ \gamma^{\text{III}}(v_+) &= (\text{defined such that } R''\gamma^{\text{III}} = -\gamma^{\text{III}})\end{aligned}$$

One can check that  $X''\gamma^{\text{III}} = -\gamma^{\text{III}}$ .

## 16.7 Parametrization of $\mathcal{P}^1/s\mathcal{P}^1$

The map

$$(\xi_a, a, \xi_b, b, \xi_c, c, d, \xi_e, e, f, g, h) \mapsto \gamma_0$$

where  $\xi_a, \xi_b, \xi_c, \xi_e \in \text{Der}(\mathcal{R})$  and  $a \in \mathcal{R}^6$ ,  $b, c \in \mathcal{R}^7$ ,  $e \in \mathcal{R}^8$ ,  $d, f, g, h \in \mathcal{R}$  and

$$\begin{aligned}\gamma_0 &= (\gamma_{\xi_a, a_1, \dots, a_5}^0 + X'\gamma_{a_6}^{\text{II}}) \\ &\quad + s'X\gamma_{\xi_b, b_1, \dots, b_7}^{\text{I}} + s''\gamma_{\xi_c, c_1, \dots, c_7}^{\text{I}} \\ &\quad + (s')^2\gamma_d^{\text{II}} + s's''(X'\gamma_{\xi_e, e_1, \dots, e_7}^{\text{I}} + X'\gamma_{e_8}^{\text{III}}) + (s'')^2X\gamma_f^{\text{II}} \\ &\quad + (s')^2s''\gamma_g^{\text{III}} + s'(s'')^2X\gamma_h^{\text{III}}\end{aligned}$$

is a parametrization of  $\mathcal{P}^1/s\mathcal{P}^1$ .

Some remarks about this parametrization:

- The first eight letters of the alphabet are used as identifiers for the components of the grading of  $\mathcal{P}^1/s\mathcal{P}^1$ , as in the following table:

$(s'')^2$	$f$	$h$	$-$
$s''$	$\xi_c, c$	$\xi_e, e$	$g$
$1$	$\xi_a, a$	$\xi_b, b$	$d$
	$1$	$s'$	$(s')^2$

- The naive leading term is  $\gamma_{(0)} = \gamma_{\xi_a, a_1, \dots, a_5}^0 + X'\gamma_{a_6}^{\text{II}}$ , consistent with §15.
- Note that

$$\begin{aligned}\gamma_0|_{\mathcal{R}} &= (v_+\overline{v_+} + v_-\overline{v_-})\xi_a \\ &\quad + s'(v_+\overline{v_-} + v_-\overline{v_+})\xi_b \\ &\quad + s''(v_+\overline{v_+} - v_-\overline{v_-})\xi_c \\ &\quad + s's''i(v_+\overline{v_-} - v_-\overline{v_+})\xi_e\end{aligned}$$

If  $\mathcal{R}$  are the smooth real functions on a 4-dim manifold, then informally, the frame  $\gamma_0|_{\mathcal{R}}$  can be nondegenerate at order  $s's''$  but no earlier.

- The way in which  $\gamma^I, \gamma^{II}, \gamma^{III}$  appear is made more transparent by throwing in an artificial third variable  $s'''$ , as discussed in §16.16.

In §16.8 ... §16.14 we explicitly write down the equation  $[\gamma_0, \gamma_0] = 0$ , organized by the grading of  $\mathcal{P}^2/s\mathcal{P}^2$ . That is, by the powers of  $s', s''$ . The bracket on  $\mathcal{P}/s\mathcal{P}$  respects this grading, hence only certain terms can appear.

### 16.8 Equation component $\mathbf{f}_{00}\mathcal{E}^2$

$$0 = -\xi_a(a_1) + (a_6)^2 + a_5a_1$$

$$0 = -\xi_a(a_2) - (a_6)^2 + a_5a_2$$

$$0 = -\xi_a(a_3) + (a_6)^2 + a_5a_3$$

$$0 = -\xi_a(a_6) + a_2a_6 + a_5a_6$$

and the equations obtained by eliminating applications of  $\xi_a$ :

$$0 = a_2a_3 + a_3a_1 + a_1a_2 - (a_6)^2$$

We already saw these equations in §15.

### 16.9 Equation component $s'(\mathbf{f}_{10}\mathcal{E}^2/\mathbf{f}_{<10}\mathcal{E}^2)$

$$0 = 2[\xi_a, \xi_b] + 2b_6\xi_a - a_1\xi_b - a_2\xi_b - 2a_5\xi_b$$

$$0 = -2\xi_a(b_1) + a_1b_1 + a_2b_1 + 2a_5b_1 + 2a_4b_6 + a_6b_6 - 2\xi_b(a_4) - \xi_b(a_6)$$

$$0 = -4\xi_a(b_2) + 2a_1b_2 + 2a_2b_2 + 4a_5b_2 + 3a_1b_6 + 3a_2b_6 + 2a_3b_6 + 4a_5b_6 - 3\xi_b(a_1) - 3\xi_b(a_2) - 2\xi_b(a_3) - 4\xi_b(a_5)$$

$$0 = 4\xi_a(b_3) - 2a_1b_3 - 2a_2b_3 - 4a_5b_3 - a_1b_6 - a_2b_6 - 2a_3b_6 - 4a_5b_6 + \xi_b(a_1) + \xi_b(a_2) + 2\xi_b(a_3) + 4\xi_b(a_5)$$

$$0 = -4\xi_a(b_4) + 2a_1b_4 + 2a_2b_4 + 4a_5b_4 - 4a_6b_5 + a_1b_6 - a_2b_6 - 4a_6b_7 - \xi_b(a_1) + \xi_b(a_2)$$

$$0 = 2\xi_a(b_5) + a_6b_2 - a_6b_3 - 2a_6b_4 - 2a_5b_5 + a_6b_6 + a_1b_7 - a_2b_7 - \xi_b(a_6)$$

and the equations obtained by eliminating applications of  $\xi_a$ :

$$0 = a_3b_2 + a_1b_3 + a_2b_3 - a_3b_3 - a_1b_4 + a_2b_4 + 2a_6b_5 + \xi_b(a_1) + \xi_b(a_2)$$

The grading only allows products of type  $ab$ .

### 16.10 Equation component $(s')^2(\mathbf{f}_{20}\mathcal{E}^2/\mathbf{f}_{<20}\mathcal{E}^2)$

$$0 = \xi_a(d) - a_1d - a_5d + \xi_b(b_5) + \xi_b(b_7) + 2b_4b_5 - b_5b_6 + 2b_4b_7 - b_6b_7$$

The grading only allows products of type  $ad, b^2$ .

### 16.11 Equation component $s's''(\mathbf{f}_{11}\mathcal{E}^2/\mathbf{f}_{<11}\mathcal{E}^2)$

$$0 = 2[\xi_a, \xi_e] + 2e_6\xi_a - a_1\xi_e - a_3\xi_e - 2a_5\xi_e + 2b_5\xi_c + 2b_7\xi_c - 2c_5\xi_b - 2c_7\xi_b$$

$$0 = -2\xi_a(e_1) + a_1e_1 + a_3e_1 + 2a_5e_1 + 2a_4e_6 + a_6e_6 + 2c_5b_1 + 2c_7b_1 - 2c_1b_5 - 2c_1b_7 - 2\xi_e(a_4) - \xi_e(a_6)$$

$$0 = -4\xi_a(e_2) + 2a_1e_2 + 2a_3e_2 + 4a_5e_2 + 3a_1e_6 + 2a_2e_6 + 3a_3e_6 + 4a_5e_6 - 2\xi_b(c_5) - 2\xi_b(c_7) + 4c_5b_2 + 4c_7b_2 - 4c_2b_5 - 2c_6b_5 + 2c_5b_6 + 2c_7b_6 - 4c_2b_7 - 2c_6b_7 + 2\xi_c(b_5) + 2\xi_c(b_7) - 3\xi_e(a_1) - 2\xi_e(a_2) - 3\xi_e(a_3) - 4\xi_e(a_5)$$

$$0 = -4\xi_a(e_3) + 2a_1e_3 + 2a_3e_3 + 4a_5e_3 + a_1e_6 + 2a_2e_6 + a_3e_6 + 4a_5e_6 + 2\xi_b(c_5) + 2\xi_b(c_7) + 4c_5b_3 + 4c_7b_3 - 4c_3b_5 + 2c_6b_5 - 2c_5b_6 - 2c_7b_6 - 4c_3b_7 + 2c_6b_7 - 2\xi_c(b_5) - 2\xi_c(b_7) - \xi_e(a_1) - 2\xi_e(a_2) - \xi_e(a_3) - 4\xi_e(a_5)$$

$$0 = 4\xi_a(e_4) - 2a_1e_4 - 2a_3e_4 - 4a_5e_4 + a_1e_6 - a_3e_6 + 2\xi_b(c_5) + 2\xi_b(c_7) + 4c_5b_4 + 4c_7b_4 - 2c_6b_5 - 2c_5b_6 - 2c_7b_6 + 4c_4b_7 - 2c_6b_7 + 2\xi_c(b_5) + 2\xi_c(b_7) - \xi_e(a_1) + \xi_e(a_3)$$

$$0 = 2\xi_a(e_5) + a_6e_2 - 3a_6e_3 + 2a_6e_4 - 2a_5e_5 - a_1e_7 + a_3e_7 + 2a_2e_8 + 2\xi_b(c_2) - \xi_b(c_3) - 3\xi_b(c_4) + \xi_b(c_6) + c_2b_2 - c_3b_2 + 2c_4b_2 + c_2b_3 - c_3b_3 - 4c_4b_3 - c_6b_3 + 2c_3b_4 - 4c_4b_4 + c_6b_4 + 2c_7b_5 - c_3b_6 + c_4b_6 - 2c_6b_6 + 2c_5b_7 + \xi_c(b_3) + \xi_c(b_4) + \xi_c(b_6) - 2\xi_e(a_6)$$

$$0 = -2\xi_a(e_8) - a_6e_2 + a_6e_3 - a_6e_6 + 2a_5e_8 - \xi_b(c_2) + \xi_b(c_4) - \xi_b(c_6) + c_3b_2 - c_4b_2 - c_2b_3 + c_4b_3 - c_6b_3 + c_2b_4 - c_3b_4 + c_6b_4 + c_3b_6 - c_4b_6 + \xi_c(b_2) - \xi_c(b_4) + \xi_c(b_6) + \xi_e(a_6)$$

and the equations obtained by eliminating applications of  $\xi_a$ :



$$\begin{aligned}
0 &= -2e_8\xi_a + a_6\xi_e + [\xi_b, \xi_c] + b_3\xi_c - b_4\xi_c - c_3\xi_b + c_4\xi_b \\
0 &= -a_6e_1 - 2a_4e_8 - a_6e_8 - \xi_b(c_1) + c_3b_1 - c_4b_1 - c_1b_3 + c_1b_4 + \xi_c(b_1) \\
0 &= -a_6e_2 - a_6e_3 - 2a_1e_8 - 2a_2e_8 - 2a_3e_8 - 4a_5e_8 - \xi_b(c_2) - \xi_b(c_3) + c_3b_2 - \\
&\quad c_4b_2 - c_2b_3 - c_4b_3 + c_2b_4 + c_3b_4 + \xi_c(b_2) + \xi_c(b_3) \\
0 &= -a_2e_2 - a_1e_3 + a_2e_3 - a_3e_3 - a_1e_4 + a_3e_4 + 2a_6e_8 - 2\xi_b(c_5) + 2c_5b_2 - \\
&\quad 4c_5b_3 - 2c_5b_4 - 2c_2b_5 + 4c_3b_5 + 2c_4b_5 + 2\xi_c(b_5) - \xi_e(a_1) - \xi_e(a_3) \\
0 &= -a_6e_2 + 3a_6e_3 - 2a_2e_8 - \xi_b(c_2) + \xi_b(c_3) + 2\xi_b(c_4) + c_3b_2 - c_4b_2 - c_2b_3 + \\
&\quad 3c_4b_3 + c_2b_4 - 3c_3b_4 + \xi_c(b_2) - \xi_c(b_3) - 2\xi_c(b_4) + 2\xi_e(a_6)
\end{aligned}$$

The grading only allows products of type  $ae, bc$ .

### 16.12 Equation component $(s')^2 s''(\mathbf{f}_{21}\mathcal{E}^2/\mathbf{f}_{<21}\mathcal{E}^2)$

$$\begin{aligned}
0 &= -4\xi_a(g) + 2a_1g + 4a_5g + \xi_b(e_2) + \xi_b(e_3) + 2\xi_b(e_6) + b_3e_2 + b_4e_2 - b_2e_3 + \\
&\quad b_4e_3 - 2b_6e_3 + b_2e_4 + b_3e_4 + 2b_6e_4 + 2b_3e_6 + 2b_4e_6 + 4b_5e_8 + 4b_7e_8 - c_2d - \\
&\quad c_3d - 2c_6d - \xi_e(b_2) - \xi_e(b_3) - 2\xi_e(b_6)
\end{aligned}$$

and the equations obtained by eliminating applications of  $\xi_a$ :

$$\begin{aligned}
0 &= -2a_4g - a_6g + \xi_b(e_1) + b_3e_1 + b_4e_1 - b_1e_3 + b_1e_4 - c_1d - \xi_e(b_1) \\
0 &= -2a_1g - 2a_2g - 2a_3g - 4a_5g + \xi_b(e_2) + \xi_b(e_3) + b_3e_2 + b_4e_2 - b_2e_3 + b_4e_3 + \\
&\quad b_2e_4 + b_3e_4 - c_2d - c_3d - \xi_e(b_2) - \xi_e(b_3) \\
0 &= -2a_1g + \xi_b(e_2) - \xi_b(e_3) - 2\xi_b(e_4) + b_3e_2 + b_4e_2 - b_2e_3 - 3b_4e_3 + b_2e_4 - \\
&\quad 3b_3e_4 - 4b_5e_8 - 4b_7e_8 + 2\xi_c(d) - c_2d + 3c_3d - \xi_e(b_2) + \xi_e(b_3) - 2\xi_e(b_4) \\
0 &= -2g\xi_a + [\xi_e, \xi_b] + e_3\xi_b - e_4\xi_b - b_3\xi_e - b_4\xi_e + d\xi_c
\end{aligned}$$

The grading only allows products of type  $ag, be, cd$ .

### 16.13 Equation component $(s')^2(s'')^2(\mathbf{f}_{22}\mathcal{E}^2/\mathbf{f}_{<22}\mathcal{E}^2)$

$$0 = \xi_b(h) + 2b_3h + b_6h + \xi_c(g) + 2c_3g + c_6g + \xi_e(e_8) + 2e_3e_8 + e_6e_8$$

The grading only allows products of type  $bh, cg, df, e^2$ .

## 16.14 Omitted equation components

We have omitted the equations components  $s'', (s'')^2, s'(s'')^2$ . They can be obtained from the analogous  $s', (s')^2, (s')^2 s''$  components.

## 16.15 Informal remark about §16.8 ... §16.14

Each of these sections contains:

- Equations involving applications of  $\xi_a$ .
- Equations obtained by eliminating applications of  $\xi_a$ .

This organization leaves considerable ambiguity, in particular equations in the 1st set can be rewritten by adding multiples of the equations in the 2nd set.

It is interesting to consider the 2nd sets of equations, assuming that  $\xi_a$  and  $a$ , which constitute the naive leading term, have already been fixed. Consider for example in §16.12 the equation:

$$[\xi_e, \xi_b] + e_3 \xi_b - e_4 \xi_b - b_3 \xi_e - b_4 \xi_e + d \xi_c = 0 \mod \mathcal{R} \xi_a$$

There are only products of type  $be, cd$  in this equation, none of type  $ag$ , which is a kind of degeneracy. See the closely related §15.4.

## 16.16 Informal remark about the parametrization in §16.7

The parametrization becomes more transparent by throwing in a third variable:

$$\begin{aligned} \gamma_0 = & \gamma_{\xi_a, a_1, \dots, a_5}^0 \\ & + s'' s''' \gamma_{\xi_c, c_1, \dots, c_7}^I + s''' s' X \gamma_{\xi_b, b_1, \dots, b_7}^I + s' s'' X' \gamma_{\xi_e, e_1, \dots, e_7}^I \\ & + (s')^2 \gamma_d^{II} + (s'')^2 X \gamma_f^{II} + (s''')^2 X' \gamma_{a_6}^{II} \\ & + s'' s''' (s')^2 \gamma_g^{III} + s''' s' (s'')^2 X \gamma_h^{III} + s' s'' (s''')^2 X' \gamma_{e_8}^{III} \end{aligned}$$

This parametrization reduces to the one in §16.7 if one sets  $s''' = 1$ .

This expression is of some interest if one is willing to speculate, with [BKL], beyond one BKL-bounce. It could conceivably be used to asymptotically match (stick together) a BKL-bounce constructed using  $s', s''$  with  $s''' = 1$  to one constructed using say  $s'', s'''$  with  $s' = 1$ . It shows which parametrization components would rotate into which other components when matching expansions.

## A Rees algebra for a $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ -indexed filtration

We elaborate on §1.11. Here  $\mathcal{P}$  satisfies, among others:

- $\mathcal{P} \rightarrow \mathcal{P}, x \mapsto s'x$  and  $\mathcal{P} \rightarrow \mathcal{P}, x \mapsto s''x$  are injective maps.
- $m_1\mathcal{P} \cap m_2\mathcal{P} \subseteq \text{lcm}(m_1, m_2)\mathcal{P}$  for any two monomials in  $s', s''$ .  
Here  $\text{lcm}$  is the least common multiple, for example  $\text{lcm}(s', s'') = s's''$ .

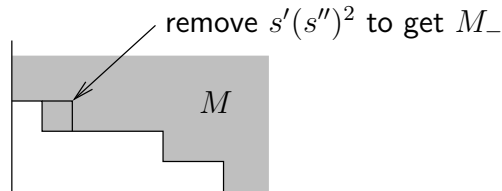
Suppose  $M$  is a set of monomials in  $s', s''$ , containing almost all monomials, and with  $s'M \subseteq M$  and  $s''M \subseteq M$ . Set  $M\mathcal{P} = \sum_{m \in M} m\mathcal{P}$ , an effectively finite sum, and an ideal of  $\mathcal{P}$ . Suppose  $M_- \subseteq M$  is such that  $s'M \subseteq M_-$  and  $s''M \subseteq M_-$ , in particular the set difference  $M \setminus M_-$  is finite. Then

$$\ker(\mathcal{P}/M\mathcal{P} \leftarrow \mathcal{P}/M_-\mathcal{P}) = M\mathcal{P}/M_-\mathcal{P} \xleftarrow{\text{vec sp iso}} \bigoplus_{m \in M \setminus M_-} \mathcal{P}/(s'\mathcal{P} + s''\mathcal{P})$$

with the isomorphism given by  $\sum_m mx_m \mapsto \bigoplus_{m \in M \setminus M_-} x_m$ .

Example 1:  $M$  the set of monomials of total degree  $\geq p$ , and  $M_-$  the set of monomials of total degree  $\geq p+1$ .

Example 2: Here  $M_-$  is obtained by removing a single monomial from  $M$ :



## B Sample evaluation of the bracket

This example uses a basis,  $V = \mathcal{C}v_- \oplus \mathcal{C}v_+$ .

**Goal.** Let  $\lambda_{\pm} \in \mathcal{C}$  be parameters. Define  $c_{\pm} \in \mathcal{L}^1$  by  $c_{\pm}(\mathcal{C}) = 0$  and by

$$\begin{aligned} c_-(v_-) &= 0 & c_+(v_-) &= \lambda_+(v_+\overline{v_-})v_+ \\ c_-(v_+) &= \lambda_-(v_-\overline{v_+})v_- & c_+(v_+) &= 0 \end{aligned}$$

Then the reality condition coming with  $\mathcal{L}^1$  implies

$$\begin{aligned} c_-(\overline{v_-}) &= 0 & c_+(\overline{v_-}) &= \overline{\lambda_+}(v_-\overline{v_+})\overline{v_+} \\ c_-(\overline{v_+}) &= \overline{\lambda_-}(v_+\overline{v_-})\overline{v_-} & c_+(\overline{v_+}) &= 0 \end{aligned}$$

Our goal is to evaluate  $\llbracket c_-, c_+ \rrbracket$ .

**Reformulation.** The above definition is equivalent to  $c_\pm = (\mathbb{1} + \mathbf{C})x_\pm$  where  $x_\pm \in \mathcal{L}_\mathbb{C}^1$  are given by  $x_\pm(\mathcal{C}) = x_\pm(\overline{V}) = 0$  and

$$\begin{aligned} x_-(v_-) &= 0 & x_+(v_-) &= \lambda_+(v_+ \overline{v_-})v_+ \\ x_-(v_+) &= \lambda_-(v_- \overline{v_+})v_- & x_+(v_+) &= 0 \end{aligned}$$

and accordingly  $(\mathbf{C}x_\pm)(\mathcal{C}) = (\mathbf{C}x_\pm)(V) = 0$  and

$$\begin{aligned} (\mathbf{C}x_-)(\overline{v_-}) &= 0 & (\mathbf{C}x_+)(\overline{v_-}) &= \overline{\lambda_+}(v_- \overline{v_+})\overline{v_+} \\ (\mathbf{C}x_-)(\overline{v_+}) &= \overline{\lambda_-}(v_+ \overline{v_-})\overline{v_-} & (\mathbf{C}x_+)(\overline{v_+}) &= 0 \end{aligned}$$

Clearly

$$\llbracket c_-, c_+ \rrbracket = \llbracket x_-, x_+ \rrbracket + \llbracket x_-, \mathbf{C}x_+ \rrbracket + \llbracket \mathbf{C}x_-, x_+ \rrbracket + \llbracket \mathbf{C}x_-, \mathbf{C}x_+ \rrbracket$$

which are four instances of the  $\mathcal{L}_\mathbb{C}^1 \times \mathcal{L}_\mathbb{C}^1 \rightarrow \mathcal{L}_\mathbb{C}^2$  bracket. We evaluate each of the four terms; only the first will require an actual calculation.

**The term  $\llbracket x_-, x_+ \rrbracket$ .** Note that  $x_\pm = \omega_\pm \delta_\pm$  where  $\omega_- = \lambda_-(v_- \overline{v_+})$  and  $\omega_+ = \lambda_+(v_+ \overline{v_-})$  and where  $\delta_\pm \in \mathcal{L}_\mathbb{C}^0$  are given by  $\delta_\pm(\mathcal{C}) = \delta_\pm(\overline{V}) = 0$  and

$$\begin{aligned} \delta_-(v_-) &= 0 & \delta_+(v_-) &= v_+ \\ \delta_-(v_+) &= v_- & \delta_+(v_+) &= 0 \end{aligned}$$

Then

$$\begin{aligned} \delta_-(\omega_+) &= \lambda_+(v_- \overline{v_-}) \\ \delta_+(\omega_-) &= \lambda_-(v_+ \overline{v_+}) \end{aligned}$$

and  $\delta = [\delta_-, \delta_+] \in \mathcal{L}_\mathbb{C}^0$  is given by  $\delta(\mathcal{C}) = \delta(\overline{V}) = 0$  and

$$\begin{aligned} \delta(v_-) &= v_- \\ \delta(v_+) &= -v_+ \end{aligned}$$

Therefore we have

$$\begin{aligned} \llbracket x_-, x_+ \rrbracket &= \llbracket \omega_- \delta_-, \omega_+ \delta_+ \rrbracket \\ &= (\omega_- \wedge \omega_+) [\delta_-, \delta_+] + (\omega_- \wedge \delta_-(\omega_+)) \delta_+ - (\delta_+(\omega_-) \wedge \omega_+) \delta_- \\ &= \lambda_- \lambda_+ [(v_- \overline{v_+} \wedge v_+ \overline{v_-}) \delta + (v_- \overline{v_+} \wedge v_- \overline{v_-}) \delta_+ - (v_+ \overline{v_+} \wedge v_+ \overline{v_-}) \delta_-] \end{aligned}$$

which is as explicit as it gets.

**The term  $\llbracket x_-, \mathbf{C}x_+ \rrbracket$ .** This term vanishes, by a calculation analogous to the one we just did. A more systematic way to see this is to introduce gradings compatible with the bracket; see §7.

**Combining.** Using  $\llbracket \mathbf{C} \cdot, \mathbf{C} \cdot \rrbracket = \mathbf{C} \llbracket \cdot, \cdot \rrbracket$  as well as  $\mathbf{C}^2 = \mathbb{1}$  we see that  $\llbracket \mathbf{C}x_-, \mathbf{C}x_+ \rrbracket = \mathbf{C} \llbracket x_-, x_+ \rrbracket$  and  $\llbracket \mathbf{C}x_-, x_+ \rrbracket = \mathbf{C} \llbracket x_-, \mathbf{C}x_+ \rrbracket = 0$ , hence

$$\llbracket c_-, c_+ \rrbracket = \llbracket x_-, x_+ \rrbracket + \mathbf{C} \llbracket x_-, x_+ \rrbracket$$

The first term annihilates  $\mathcal{C} \oplus \overline{V}$ , the second annihilates  $\mathcal{C} \oplus V$ . One can think of it like this:  $\llbracket c_-, c_+ \rrbracket|_V = \llbracket x_-, x_+ \rrbracket|_V$ , and then  $\llbracket c_-, c_+ \rrbracket|_{\overline{V}}$  is determined by reality.

**Final observation.** Set  $x' = \lambda_- \lambda_+ (v_- \overline{v_-} \wedge v_+ \overline{v_+}) \delta \in \mathcal{L}_{\mathbb{C}}^2$  and add-subtract:

$$\llbracket x_-, x_+ \rrbracket = x' + (-x' + \llbracket x_-, x_+ \rrbracket)$$

One verifies that the second term is in  $\mathcal{I}^2 \oplus i\mathcal{I}^2$ , hence

$$\llbracket c_-, c_+ \rrbracket \in (\mathbb{1} + \mathbf{C})x' + \mathcal{I}^2$$

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